8 Post-Newtonian theory: Fundamentals

Post-Newtonian theory is the theory of weak-field gravity within the near zone, and of the slowly moving systems that generate it and respond to it. It was first encountered in Chapter 7, where it was embedded within the post-Minkowskian approximation; the idea relies on the slow-motion condition introduced in Sec. 6.3.2. But while post-Minkowskian theory deals with both the near and wave zone, here we focus exclusively on the near zone. In this chapter we develop the post-Newtonian theory systematically.

We begin in Sec. 8.1 by collecting the main ingredients obtained in Chapter 7, including the near-zone metric to 1PN order and the matter's energy-momentum tensor $T^{\alpha\beta}$. In Sec. 8.2 we present an alternative derivation of the post-Newtonian metric, based on the Einstein equations in their standard form; this is the "classic approach" to post-Newtonian theory, adopted by Einstein, Infeld, and Hoffmann in the 1930s, and by Fock, Chandrasekhar, and others in the 1960s. Although it produces the same results, we will see that the classic approach presents us with a number of ambiguities that are *not* present in the post-Minkowskian approach. In Sec. 8.3 we explore the coordinate freedom of post-Newtonian theory, and construct the most general transformation that preserves the post-Newtonian expansion of the metric. And in Sec. 8.4 we derive the laws of fluid dynamics in post-Newtonian theory; these will be applied to the motion of an N-body system in Chapter 9.

8.1 Equations of post-Newtonian theory

8.1.1 Post-Newtonian metric

We restrict our attention to a matter distribution that is subjected to a *slow-motion condition* of the sort first considered in Sec. 6.3.2. The distribution is characterized by a length scale r_c and a time scale t_c, and these give us the characteristic velocity $v_c := r_c/t_c$. We assume that this is much smaller than the speed of light,

$$v_c/c \ll 1, \tag{8.1}$$

and this defines what we mean by the slow-motion condition: all speeds within the matter distribution (such as the speed of sound within a body, or the speed of the body as a whole) shall be small compared with the speed of light. If $\lambda_c := ct_c$ is a characteristic wavelength of the gravitational radiation produced by the matter distribution, then Eq. (8.1) states that $r_c \ll \lambda_c$. The region of space occupied by the matter is therefore small compared with

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the characteristic wavelength; the matter is situated deep within the *near-zone region* of spacetime, defined by $r := |\boldsymbol{x}| \ll \lambda_c$.

Incorporating these assumptions, focusing on field points within the near zone, and carrying out two iterations of the relaxed field equations, we obtained the spacetime metric of a post-Newtonian system back in Sec. 7.3. The metric is displayed in Eq. (7.105), and we reproduce it here:

$$g_{00} = -1 + \frac{2}{c^2}U + \frac{2}{c^4}\left(\Psi - U^2\right) + O(c^{-6}), \tag{8.2a}$$

$$g_{0j} = -\frac{4}{c^3}U_j + O(c^{-5}), \tag{8.2b}$$

$$g_{jk} = \delta_{jk}\left(1 + \frac{2}{c^2}U\right) + O(c^{-4}), \tag{8.2c}$$

where

$$\Psi := \psi + \frac{1}{2}\partial_{tt}X. \tag{8.3}$$

The potentials that appear in the metric are defined by

$$U(t, \boldsymbol{x}) := G\int \frac{\rho^{*\prime}}{|\boldsymbol{x} - \boldsymbol{x}'|}\, d^3x', \tag{8.4a}$$

$$\psi(t, \boldsymbol{x}) := G\int \frac{\rho^{*\prime}\left(\frac{3}{2}v'^2 - U' + \Pi' + 3p'/\rho^{*\prime}\right)}{|\boldsymbol{x} - \boldsymbol{x}'|}\, d^3x', \tag{8.4b}$$

$$X(t, \boldsymbol{x}) := G\int \rho^{*\prime}|\boldsymbol{x} - \boldsymbol{x}'|\, d^3x', \tag{8.4c}$$

$$U^j(t, \boldsymbol{x}) := G\int \frac{\rho^{*\prime}v'^j}{|\boldsymbol{x} - \boldsymbol{x}'|}\, d^3x', \tag{8.4d}$$

in which the primed fluid variables are evaluated at time t and position $\boldsymbol{x}'$; these are determined by the equations of fluid dynamics to be derived in Sec. 8.4. As in the Newtonian theory, the dynamics of the fluid and the dynamics of the gravitational field are intimately coupled to each other. It should be noted that the potentials of Eqs. (8.4) are all *instantaneous potentials*: their profile at time t depends on the state of the system at the same time. The metric, however, does incorporate retardation effects that arise from solving the wave equation for the gravitational potentials $h^{\alpha\beta}$; these are captured by the superpotential term $\partial_{tt}X$ in g_{00}, which appears when h^{00} is expanded in powers of c^{-2} within the near zone.

The post-Newtonian metric makes a good approximation to the true spacetime metric in the near zone only; the approximation is not valid beyond $r = \lambda_c$. The reason for this limitation has already been invoked in Box 6.6. It has to do with the fact that while the behavior of the metric in the near zone is directly tied to the behavior of the matter, so that the metric varies slowly when the matter moves slowly, this is not so in the wave zone, where the radiative behavior of the metric asserts itself. Mathematically, the slow behavior of the metric in the near zone is expressed by the equation

$$\partial_0 g_{\alpha\beta} \sim \frac{v_c}{c}\,\partial_j g_{\alpha\beta}, \tag{8.5}$$

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which states that derivatives with respect to $x^0 := ct$ are smaller than spatial derivatives by a factor of order $v_c/c \ll 1$. If we imagine, for example, a matter distribution that consists of N isolated bodies with positions $\boldsymbol{r}_A(t)$, then the metric will depend on time through the N position vectors, and temporal derivatives will be generated by spatial differentiation followed by differentiation of $\boldsymbol{r}_A(t)$ with respect to time; and we see that these operations do indeed bring out the additional factors of v_c/c. The situation is very different in the wave zone, because of the radiative nature of the metric when $r > \lambda_c$. Here the characteristic velocity of the field becomes the speed of light, and it is no longer related to the matter's velocity scale. As a result, $\partial_0 g_{\alpha\beta}$ is of the same order of magnitude as the spatial derivatives, and the slow-motion condition no longer has the same effect on the behavior of the metric.

8.1.2 Energy-momentum tensor

The metric of Eq. (8.2) was constructed from potentials $h_2^{\alpha\beta}$ obtained after two iterations of the relaxed Einstein equations. These potentials can then be involved in a computation of $\tau_2^{\alpha\beta}$, the effective energy-momentum pseudotensor, which can be substituted into the conservation statement $\partial_\beta \tau_2^{\alpha\beta} = 0$ to obtain the system's equations of motion. But since this conservation statement is formally equivalent to the covariant expression of energy-momentum conservation, $\nabla_\beta T^{\alpha\beta} = 0$, an alternative method to obtain the equations of motion is to compute $T^{\alpha\beta}$ to the required degree of accuracy, and to insert it within the covariant equation. This alternative method turns out to be simpler to implement than the original one involving $\tau_2^{\alpha\beta}$.

This program will be implemented in Sec. 8.4. In preparation for this discussion, we now compute $T^{\alpha\beta}$ to the required post-Newtonian order. We recall that

$$T^{\alpha\beta} = (\rho + \epsilon/c^2 + p/c^2)u^\alpha u^\beta + p g^{\alpha\beta}\,, \tag{8.6}$$

that $\rho = \rho^*(1 - v^2/2c^2 - 3U/c^2) + O(c^{-4})$ and that $u^\alpha = \gamma(c, \boldsymbol{v})$, with $\gamma := u^0/c = 1 + v^2/2c^2 + U/c^2 + O(c^{-4})$. These relations give us

$$c^{-2}T^{00} = \rho^*\biggl[1 + \frac{1}{c^2}\Bigl(\frac{1}{2}v^2 - U + \Pi\Bigr)\biggr] + O(c^{-4})\,, \tag{8.7a}$$

$$c^{-1}T^{0j} = \rho^* v^j\biggl[1 + \frac{1}{c^2}\Bigl(\frac{1}{2}v^2 - U + \Pi + p/\rho^*\Bigr)\biggr] + O(c^{-4})\,, \tag{8.7b}$$

$$\begin{aligned} T^{jk} &= \rho^* v^j v^k\biggl[1 + \frac{1}{c^2}\Bigl(\frac{1}{2}v^2 - U + \Pi + p/\rho^*\Bigr)\biggr] + p\Bigl(1 - \frac{2}{c^2}U\Bigr)\delta^{jk} \\ &\quad + O(c^{-4})\,, \end{aligned} \tag{8.7c}$$

where $\Pi := \epsilon/\rho^*$. Note that T^{00} is expanded to order c^0, T^{0j} to order c^{-1}, and T^{jk} to order c^{-2}. The metric, on the other hand, is expanded to order c^{-4} for g_{00}, c^{-3} for g_{0j}, and c^{-2} for g_{jk}. As expressed here, the components of the energy-momentum tensor follow a reversed hierarchy of post-Newtonian orders compared to the components of the metric. The reason for this is that each component of the energy-momentum tensor must contain a leading-order piece and a post-Newtonian correction if it is to yield useful information at 1PN order. This is because the components will be inserted within the conservation equations, which

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take the schematic form $c^{-1}\partial_t T^{\alpha 0} \approx -\partial_k T^{\alpha k}$; while the leading-order pieces will deliver the system's Newtonian dynamics, it is the post-Newtonian corrections that will deliver the post-Newtonian dynamics.

8.1.3 Auxiliary potentials

For future reference we list here a number of post-Newtonian potentials that can also be associated with a perfect fluid; some appear in the potential ψ, while others will be used later in this chapter:

$$\Phi_1 := G \int \frac{\rho^{*\prime} v^{\prime 2}}{|\boldsymbol{x} - \boldsymbol{x}'|}\, d^3x' , \tag{8.8a}$$

$$\Phi_2 := G \int \frac{\rho^{*\prime} U'}{|\boldsymbol{x} - \boldsymbol{x}'|}\, d^3x' , \tag{8.8b}$$

$$\Phi_3 := G \int \frac{\rho^{*\prime} \Pi'}{|\boldsymbol{x} - \boldsymbol{x}'|}\, d^3x' , \tag{8.8c}$$

$$\Phi_4 := G \int \frac{p'}{|\boldsymbol{x} - \boldsymbol{x}'|}\, d^3x' , \tag{8.8d}$$

$$\Phi_5 := G \int \rho^{*\prime} \partial_{j'} U' \frac{(x - x')^j}{|\boldsymbol{x} - \boldsymbol{x}'|}\, d^3x' , \tag{8.8e}$$

$$\Phi_6 := G \int \rho^{*\prime} v'_j v'_k \frac{(x - x')^j (x - x')^k}{|\boldsymbol{x} - \boldsymbol{x}'|^3}\, d^3x' , \tag{8.8f}$$

$$\Phi^j := G \int \rho^{*\prime} v'_k \frac{(x - x')^j (x - x')^k}{|\boldsymbol{x} - \boldsymbol{x}'|^3}\, d^3x' . \tag{8.8g}$$

Again a primed variable such as $\rho^{*\prime}$ stands for $\rho^*(t, \boldsymbol{x}')$, and $\partial_{j'} U'$ stands for the partial derivative of $U(t, \boldsymbol{x}')$ with respect to x'^j.

Referring to Eq. (8.4b), we see immediately that

$$\psi = \frac{3}{2}\Phi_1 - \Phi_2 + \Phi_3 + 3\Phi_4 . \tag{8.9}$$

In Sec. 8.4.4 we shall have occasion to prove that

$$\partial_{tj} X = \Phi_j - U_j , \tag{8.10a}$$

$$\partial_{tt} X = \Phi_1 + 2\Phi_4 - \Phi_5 - \Phi_6 . \tag{8.10b}$$

Another useful identity is

$$\partial_{jk} X = \delta_{jk} U - G \int \rho^{*\prime} \frac{(x - x')^j (x - x')^k}{|\boldsymbol{x} - \boldsymbol{x}'|^3}\, d^3x' . \tag{8.11}$$

This equation follows directly from the definition of the superpotential, and taking its trace confirms that $\nabla^2 X = 2U$. Combining Eqs. (8.3), (8.9), and (8.10b), we arrive at

$$\Psi = 2\Phi_1 - \Phi_2 + \Phi_3 + 4\Phi_4 - \frac{1}{2}\Phi_5 - \frac{1}{2}\Phi_6, \tag{8.12}$$

a useful decomposition of the post-Newtonian potential Ψ in terms of the auxiliary potentials.

8.1.4 Geodesic equations

To conclude this section we derive the form of the geodesic equation that governs the motion of a test particle in the post-Newtonian spacetime. We examine both the case of a test body that moves slowly ($v/c \ll 1$), and the case of a massless particle (such as a photon) that moves rapidly ($v/c \simeq 1$).

In either case the geodesic equation is

$$\frac{d^2 r^\alpha}{d\lambda^2} + \Gamma^\alpha_{\beta\gamma} \frac{dr^\beta}{d\lambda} \frac{dr^\gamma}{d\lambda} = 0, \tag{8.13}$$

in which $r^\alpha(\lambda)$ describes the particle's world line in spacetime; the parameter λ is proper time τ in the case of a massive body, and an arbitrary affine parameter in the case of a photon. For our purposes it is useful to alter the parameterization of the world line and adopt the time coordinate $t := x^0/c$ instead of λ. There is a practical reason for this change: the motion of a planet or spacecraft, or the trajectory of a light ray in space, is generally tracked by an external observer who employs a clock that measures external time t instead of the planet's proper time λ; a description of the motion in terms of t is therefore much more useful to this observer. (We shall return to this theme in Chapter 10, and give a more precise description of the relation between t and the observer's clock time.) A straightforward application of the chain rule reveals that the geodesic equation becomes

$$\frac{dv^\alpha}{dt} = -\left(\Gamma^\alpha_{\beta\gamma} - \frac{v^\alpha}{c} \Gamma^0_{\beta\gamma} \right) v^\beta v^\gamma \tag{8.14}$$

when the world line is parameterized by t; here $v^\alpha := dr^\alpha/dt = (c, \boldsymbol{v})$. The time component of Eq. (8.14) returns $0 = 0$, and the motion of the particle is completely determined by the spatial components.

The Christoffel symbols required for the geodesic equation are obtained from Eq. (8.2), which we insert into Eq. (5.34). We get

$$\Gamma^0_{00} = -\frac{1}{c^3} \partial_t U + O(c^{-5}), \tag{8.15a}$$

$$\Gamma^0_{0j} = -\frac{1}{c^2} \partial_j U + O(c^{-4}), \tag{8.15b}$$

$$\Gamma^0_{jk} = \frac{2}{c^3} \big(\partial_j U_k + \partial_k U_j \big) + \frac{1}{c^3} \delta_{jk} \partial_t U + O(c^{-5}), \tag{8.15c}$$

$$\Gamma^j_{00} = -\frac{1}{c^2} \partial_j U - \frac{1}{c^4} \big(4\partial_t U_j + \partial_j \Psi - 4U \partial_j U \big) + O(c^{-6}), \tag{8.15d}$$

$$\Gamma^j_{0k} = \frac{1}{c^3} \delta_{jk} \partial_t U - \frac{2}{c^3} \big(\partial_k U_j - \partial_j U_k \big) + O(c^{-5}), \tag{8.15e}$$

$$\Gamma^j_{kn} = \frac{1}{c^2} \big(\delta_{jn} \partial_k U + \delta_{jk} \partial_n U - \delta_{kn} \partial_j U \big) + O(c^{-4}), \tag{8.15f}$$

and making the substitution within Eq. (8.14), we obtain

$$\begin{aligned} \frac{dv^j}{dt} = \partial_j U + \frac{1}{c^2} \Big[& (v^2 - 4U) \partial_j U - \big(4v^k \partial_k U + 3\partial_t U \big) v^j \\ & - 4v^k \big(\partial_j U_k - \partial_k U_j \big) + 4\partial_t U_j + \partial_j \Psi \Big] + O(c^{-4}) \end{aligned} \tag{8.16}$$

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when the particle is a massive body that moves slowly, so that $v/c \ll 1$. In the first term we recognize the Newtonian acceleration field $\partial_j U$, and the remaining terms are post-Newtonian corrections. The 1PN terms are suppressed by factors of order $(v/c)^2$, vv_c/c^2, or $(v_c/c)^2$, where v is the magnitude of the particle's velocity, while v_c is the characteristic velocity scale of the matter distribution. For example $\partial_t U$ is of order v_c relative to $\partial_j U$, and this is multiplied by v^j/c^2 in the equations of motion; the contribution is therefore of order vv_c/c^2.

In the case of a photon we cannot take v to be much smaller than c, and the geodesic equation permits an expansion in powers of v_c/c only. The magnitude of $\boldsymbol{v}$ can be determined from the lightlike condition $g_{\alpha\beta}v^\alpha v^\beta = 0$. To leading order in the post-Newtonian expansion we find that $(v/c)^2 = 1 - 4U/c^2$, and this relation neglects terms of order $v_c U/c^3$. This implies that $\boldsymbol{v}$ can be expressed as

$$\boldsymbol{v} = c\Big(1 - \frac{2}{c^2}U\Big)\boldsymbol{n} + O(c^{-3}), \tag{8.17}$$

in terms of a unit vector $\boldsymbol{n}$. This equation reveals that the coordinate velocity of a photon deviates from c in curved spacetime. If we take v/c to be of order unity but continue to treat v_c/c as a small quantity, Eq. (8.14) produces

$$\frac{dv^j}{dt} = \bigg(1 + \frac{v^2}{c^2}\bigg)\partial_j U - \frac{4}{c^2}v^j v^k \partial_k U + O(c^{-3}). \tag{8.18}$$

The geodesic equation becomes

$$\frac{dn^j}{dt} = \frac{2}{c}\big(\delta^{jk} - n^j n^k\big)\partial_k U + O(c^{-2}) \tag{8.19}$$

after making the substitution of Eq. (8.17). We note that the right-hand side of Eq. (8.19) is orthogonal to n_j; this is as it should be, because $n_j dn^j/dt = \frac{1}{2}d(n_j n^j)/dt = 0$.

Box 8.1 **Maxwell-like formulation of post-Newtonian theory**

The main equations of post-Newtonian theory can be written in a form that displays a remarkable parallel with the equations of electrodynamics. These consist of Maxwell's equations,

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\boldsymbol{E} &= \frac{1}{\epsilon_0}\rho_e,\\
\boldsymbol{\nabla}\cdot\boldsymbol{B} &= 0,\\
\boldsymbol{\nabla}\times\boldsymbol{E} &= -\partial_t\boldsymbol{B},\\
\boldsymbol{\nabla}\times\boldsymbol{B} &= \frac{1}{c^2}\bigg(\frac{1}{\epsilon_0}\boldsymbol{j}_e + \partial_t\boldsymbol{E}\bigg),
\end{aligned}$$

which govern the behavior of the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$ in terms of the charge density $\rho_e = c^{-1}j_e^0$ and the current density $\boldsymbol{j}_e$, and the Lorentz-force law

$$m\frac{d(\gamma\boldsymbol{v})}{dt} = q\big(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}\big),$$

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with $\gamma = dt/d\tau$, which governs the behavior of a particle of mass m, charge q, and velocity $\boldsymbol{v}$ in the electromagnetic field.

From the post-Newtonian metric (8.2), we first define a gravito-electric potential $\Phi_{\rm g} := -\frac{1}{2}c^2(1 + g_{00}) = -U - c^{-2}(\Phi - U^2) + O(c^{-4})$ and a gravito-magnetic potential $\boldsymbol{A}_{\rm g}$ with components $cg_{0j} = -4c^{-2}U_j + O(c^{-4})$. We next define a gravito-electric field $\boldsymbol{E}_{\rm g} := -\boldsymbol{\nabla}\Phi_{\rm g} - \partial_t \boldsymbol{A}_g = \boldsymbol{\nabla} U + c^{-2}(\boldsymbol{\nabla}\Psi - \boldsymbol{\nabla} U^2 + 4\partial_t \boldsymbol{U})$ and a gravito-magnetic field $\boldsymbol{B}_{\rm g} := \boldsymbol{\nabla} \times \boldsymbol{A}_{\rm g} = -4c^{-2}\boldsymbol{\nabla} \times \boldsymbol{U}$; the relations between potentials and fields are the same as in electrodynamics. It is then a simple matter to show that the field equations of post-Newtonian theory can be put in the Maxwell-like form

$$\begin{aligned}
\tilde{\boldsymbol{\nabla}} \cdot \boldsymbol{E}_{\rm g} &= -4\pi G\rho^*\left[1 + \frac{1}{c^2}\left(\frac{3}{2}v^2 - 3U + \Pi + 3p/\rho^*\right)\right] - \frac{3}{c^2}\partial_{tt}U + O(c^{-4}), \\
\boldsymbol{\nabla} \cdot \boldsymbol{B}_{\rm g} &= O(c^{-4}), \\
\boldsymbol{\nabla} \times \boldsymbol{E}_{\rm g} &= -\partial_t \boldsymbol{B}_{\rm g}, \\
\boldsymbol{\nabla} \times \boldsymbol{B}_{\rm g} &= \frac{4}{c^2}\left(-4\pi G\rho^* \boldsymbol{v} + \partial_t \boldsymbol{E}_{\rm g}\right) + O(c^{-4}),
\end{aligned}$$

where $\tilde{\boldsymbol{\nabla}} \cdot \boldsymbol{E}_{\rm g}$ denotes a "curved-space" divergence $(-g)^{-1/2}\boldsymbol{\nabla}[(-g)^{1/2}\boldsymbol{E}_{\rm g}]$, where $-g = 1 + 2U/c^2 + O(c^{-4})$ is the determinant of the post-Newtonian metric. It is also simple to show that the geodesic equation acquires a Lorentz-like form

$$\frac{d(g_s \gamma \boldsymbol{v})}{dt} = \gamma\left(\boldsymbol{E}_{\rm g} + \boldsymbol{v} \times \boldsymbol{B}_{\rm g} + v^2 \boldsymbol{\nabla} g_s\right) + O(c^{-4}), \qquad (1)$$

where $g_s := 1 + 2U/c^2$ is the coefficient of the spatial part of the PN metric, and $\gamma := dt/d\tau = 1 + c^{-2}(\frac{1}{2}v^2 + U) + O(c^{-4})$. This can be expressed in the more explicit form

$$\frac{d}{dt}\left\{\left[1 + \frac{1}{c^2}\left(\frac{1}{2}v^2 + 3U\right)\right]\boldsymbol{v}\right\} = \left[1 + \frac{1}{c^2}\left(\frac{3}{2}v^2 - U\right)\right]\boldsymbol{E}_{\rm g} + \boldsymbol{v} \times \boldsymbol{B}_{\rm g} + O(c^{-4}).$$

Apart from additional post-Newtonian terms, the equations are indeed remarkably similar to those of the Maxwell–Lorentz theory, with ρ^* playing the role of the charge density, $\rho^* \boldsymbol{v}$ that of the current density, and $-4\pi G$ playing the role of the coupling constant $1/\epsilon_0$.

There are, however, clear indications that gravity is different from electrodynamics. Apart from the additional post-Newtonian terms, the most important differences are seen in the sign of the coupling constant and the factor of 4 in the $\boldsymbol{\nabla} \times \boldsymbol{B}_{\rm g}$ equation. The gravitational coupling constant $-4\pi G$ is negative instead of positive, reflecting the fact that in gravity, like charges attract instead of repel. The factor of 4 reminds us that the gravitational potentials $\Phi_{\rm g}$ and $\boldsymbol{A}_{\rm g}$ originate from a tensor (the metric) instead of a vector; in quantum parlance we say that the graviton is a spin-2 particle, while the photon is a spin-1 particle. The Lorentz-like equation would be identical to that of electrodynamics through $O(c^{-2})$ were it not for the appearance of the spatial part of the metric, represented by the factor g_s and the extra factor of γ on the right-hand-side of Eq. (1), reflecting its true origin in the geodesic equation.

Nevertheless, the Maxwell-like formulation of the post-Newtonian approximation to Einstein's equations and the geodesic equation can be very useful in specific situations, particularly when some of the additional post-Newtonian terms can be neglected. This occurs, for example, when the fields are stationary, when non-linear contributions (proportional to $\rho^* U$) can be ignored, or when the velocity field inside the source is particularly small. In such cases, solutions to the post-Newtonian equations can be imported with minimal modifications from electrodynamics, along with the attached intuition. The Maxwell-like formulation has been used to study everything from laboratory and space experiments to test general relativity to the behavior of matter around rotating black holes.

8.2 Classic approach to post-Newtonian theory

Before we proceed with our exploration of post-Newtonian theory, it is instructive to provide an alternative derivation of the metric based on the standard formulation of the Einstein field equations instead of the Landau–Lifshitz formulation reviewed in Chapter 6. We refer to this derivation as the *classic approach* to post-Newtonian theory, and our quick survey will reveal some of the ambiguities and conceptual difficulties associated with it. The *modern approach* to post-Newtonian theory, based on its post-Minkowskian foundation, is completely free of such ambiguities and conceptual difficulties.

We begin by *postulating* a form of the metric to 1PN order:

$$g_{00} = -1 + \frac{2}{c^2}U + \frac{2}{c^4}\big(\Psi - U^2\big) + O(c^{-6}), \tag{8.20a}$$

$$g_{0j} = -\frac{4}{c^3}U_j + O(c^{-5}), \tag{8.20b}$$

$$g_{jk} = \left(1 + \frac{2}{c^2}U\right)\delta_{jk} + O(c^{-4}), \tag{8.20c}$$

where U, U^j, and Ψ are gravitational potentials to be determined. The term of order c^{-2} in g_{00} is a Newtonian term. The terms of order c^{-4} in g_{00}, c^{-3} in g_{0j}, and c^{-2} in g_{jk} are post-Newtonian terms. The insertion of U^2 within g_{00} simplifies the form of the field equations. A blind post-Newtonian expansion of g_{jk} would introduce a general tensorial potential U_{jk} instead of the specific expression $U\delta_{jk}$ that involves the same potential U as in g_{00}. To keep the algebra simple, we anticipate the result of an integration of the Einstein field equations at lowest order, which reveals that indeed, U_{jk} must be equal to $U\delta_{jk}$. (In fact, we have reached this conclusion already in Sec. 5.5, when we studied the linearized approximation of general relativity.) We impose a harmonic coordinate condition, as displayed in Eq. (6.47); this reduces to

$$\partial_t U + \partial_j U^j = 0 \tag{8.21}$$

at the required post-Newtonian order. Again we state that at the outset, U, U^j, and Ψ are unknown functions to be determined by the field equations; apart from our assumption regarding the tensorial potential, there is no loss of generality in Eq. (8.20).

The standard formulation of the Einstein field equations is $G_{\alpha\beta} = (8\pi G/c^4)T_{\alpha\beta}$, and recalling the definition of the Einstein tensor from Eq. (5.71), this is

$$R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = \frac{8\pi G}{c^4}T_{\alpha\beta}, \tag{8.22}$$

in which $R_{\alpha\beta}$ is the Ricci tensor and $R := g^{\alpha\beta}R_{\alpha\beta}$ the Ricci scalar. Taking the trace yields $R = -(8\pi G/c^4)T$, in which $T := g^{\alpha\beta}R_{\alpha\beta}$, and making the substitution back in the field equations produces

$$R_{\alpha\beta} = \frac{8\pi G}{c^4}\bar{T}_{\alpha\beta}, \tag{8.23}$$

in which $\bar{T}_{\alpha\beta} := T_{\alpha\beta} - \frac{1}{2}Tg_{\alpha\beta}$ is the "trace-reversed" energy-momentum tensor. This form of the field equations is our starting point for the determination of the potentials U, U^j, and Ψ. It is advantageous because the computation of the Ricci tensor from the metric of Eq. (8.20) is relatively straightforward. The computation of the Einstein tensor would require additional steps and make the entire task more tedious.

A straightforward calculation using the Christoffel symbols of Eqs. (8.15) reveals that the components of the Ricci tensor are

$$R_{00} = -\frac{1}{c^2}\nabla^2 U + \frac{1}{c^4}\big(\partial_{tt}U + 4U\nabla^2 U - \nabla^2\Psi\big) + O(c^{-6}), \tag{8.24a}$$

$$R_{0j} = \frac{2}{c^3}\nabla^2 U_j + O(c^{-5}), \tag{8.24b}$$

$$R_{jk} = -\frac{1}{c^2}\nabla^2 U\delta_{jk} + O(c^{-4}). \tag{8.24c}$$

We have used Eq. (8.21) to eliminate terms involving $\partial_j U^j$ in favor of terms involving $\partial_t U$.

Importing the components of the energy-momentum tensor from Eq. (8.7), we have that

$$T_{00} = \rho^* c^2\left[1 + \frac{1}{c^2}\left(\frac{1}{2}v^2 - 5U + \Pi\right)\right] + O(c^{-2}), \tag{8.25a}$$

$$T_{0j} = -\rho^* v^j c + O(c^{-1}), \tag{8.25b}$$

$$T_{jk} = \rho^* v^j v^k + p\,\delta^{jk} + O(c^{-2}), \tag{8.25c}$$

to the required post-Newtonian order. To leading order this produces $\bar{T}_{00} = \frac{1}{2}\rho^* c^2 + O(1)$, and the c^{-2} piece of the 00 component of the field equations yields

$$\nabla^2 U = -4\pi G\rho^*. \tag{8.26}$$

We conclude that as expected, U is the standard Newtonian potential defined by Eq. (8.4). Also to leading order, $\bar{T}_{jk} = \frac{1}{2}\delta_{jk}\rho^* c^2 + O(1)$, and we see that the c^{-2} piece of the jk components of the field equations is automatically satisfied; this validates our assumed form for the spatial part of the metric. The c^{-3} piece of the $0j$ components of the field equations yields

$$\nabla^2 U_j = -4\pi G\rho^* v_j, \tag{8.27}$$

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and U_j is the standard vector potential as defined by Eq. (8.4). Finally, the c^{-4} piece of the 00 component of the field equations yields

$$\partial_{tt}U + 4U\nabla^2 U - \nabla^2\Psi = 4\pi G\rho^*\left(\frac{3}{2}v^2 - 5U + \Pi + \frac{3p}{\rho^*}\right). \tag{8.28}$$

This is a Poisson equation for Ψ, and after substituting $\nabla^2 U = -4\pi G\rho^*$ on the left-hand side and rearranging, we see that Ψ is given by

$$\Psi = \psi + \frac{1}{2}\partial_{tt}F, \tag{8.29}$$

with ψ and F required to satisfy

$$\nabla^2\psi = -4\pi G\rho^*\left(\frac{3}{2}v^2 - U + \Pi + \frac{3p}{\rho^*}\right), \tag{8.30}$$

and

$$\nabla^2 F = 2U. \tag{8.31}$$

The solution to Eq. (8.30) is evidently the potential defined by Eq. (8.4b).

To identify the solution to Eq. (8.31) requires a more careful discussion, because the source term $2U$ is not limited to the region occupied by the matter distribution; it is distributed over all space. This discussion has already been provided in Box. 7.3, where it was shown that the general solution is given by

$$F = X - 2m\mathcal{R} + F_0, \tag{8.32}$$

in which $X(t, \boldsymbol{x})$ is the standard superpotential as displayed in Eq. (8.4), $m := \int \rho^*\, d^3x$ is the total rest-mass of the fluid system, $\mathcal{R}$ is a constant length, and $F_0(t, \boldsymbol{x})$ is a solution to Laplace's equation. Demanding that F_0 does not depend on time implies that $\partial_{tt}F = \partial_{tt}X$, so that $\Psi = \psi + \partial_{tt}X$ as required by Eq. (8.3). Choosing $F_0 = 2m\mathcal{R}$ returns the stronger equality $F = X$, yielding the same expression for Ψ.

The preceding discussion indicates that when appropriate choices are made, the solution to Eq. (8.31) returns the correct expression for Ψ, as given by Eq. (8.3). The main question that arises is: which guiding principle can be invoked to justify the choices made to specify this solution? The answer is simply that no such principle exists within the strict context of the classic approach to post-Newtonian theory; the ambiguities associated with F can only be resolved with ad hoc choices. For a more satisfying resolution, one must turn to the modern approach and its post-Minkowskian foundation.

In the classic approach, the superpotential arises as a particular solution to $\nabla^2 F = 2U$, and the choice of solution is ambiguous because U extends over all space. (The potentials U, U^j, and ψ do not share this ambiguity, because their source terms are tied to the matter distribution.) In the modern approach, the superpotential arises in an expansion of h^{00} in powers of c^{-1} in the near zone, and it is fundamentally defined as the integral $\int \rho^{*\prime}|\boldsymbol{x} - \boldsymbol{x}'|\, d^3x'$; the Poisson equation follows as a consequence of this definition. The advantages of the modern approach should be clear. First, the post-Minkowskian foundation provides a clear restriction of the post-Newtonian metric to the near zone,

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while no such restriction is immediately apparent in the classic approach. Second, while the classic approach features Poisson equations with ambiguous solutions, the modern approach defines all potentials in terms of near-zone integrals that are devoid of ambiguities.

A third advantage is concerned with the incorporation of retardation effects in the post-Newtonian metric. Our experience with post-Minkowskian theory allows us to locate the retardation in the $\partial_{tt} X$ term, but we would be hard pressed to provide this understanding if we were only familiar with the classic approach. Indeed, because the classic approach defines each potential in terms of a Poisson equation, each potential will necessarily be instantaneous, and the retardation effects will be implicit and hidden from view. There is actually a deeper problem that is revealed at higher post-Newtonian orders. A systematic development of the classic approach to higher orders would continue to introduce potentials that satisfy Poisson equations, and the ambiguities would pile up. In particular, it would quickly become unclear how to impose a condition that the metric should describe *outgoing* gravitational waves at infinity. Because the post-Newtonian expansion is necessarily limited to the near-zone region of space, the wave zone is inaccessible, and the boundary conditions cannot be formulated in a clean way. In the modern approach, the post-Minkowskian formulation of the problem is based on wave equations instead of Poisson equations, and the selection of retarded solutions ensures that the waves are properly outgoing in the wave zone. It is this specific choice of solution that provides the retardation and makes the near-zone metric completely unambiguous.

In his sequence of papers on post-Newtonian hydrodynamics written between 1964 and 1969, Chandrasekhar employed the classic approach outlined in this section to derive the metric and the equations of motion. Working with his student Yavuz Nutku, he continued to employ this method to obtain the 2PN equations of hydrodynamics. But when it came time in 1970 to move on to the 2.5PN equations of motion, an order at which the selection of outgoing-wave boundary conditions is essential, Chandra and his student Paul Esposito finally recognized the limitations described here. They converted to the modern approach.

8.3 Coordinate transformations

8.3.1 Introduction

In this section we explore the freedom that post-Newtonian theory possesses to perform coordinate transformations that preserve the post-Newtonian ordering of the metric. We wish to find the most general class of transformations

$$t = t(\bar{t}, \bar{x}^j), \qquad x^j = x^j(\bar{t}, \bar{x}^j), \tag{8.33}$$

that keeps the metric expressed as an expansion in powers of c^{-2}. We call these *post-Newtonian transformations*, and construct them step-by-step in Secs. 8.3.2 and 8.3.3.

We shall find that in general, the post-Newtonian transformations do not preserve the harmonic coordinate condition of Eq. (8.21). In Sec. 8.3.4 we specialize them to a class that

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keeps the coordinates harmonic; we call this restricted class the *harmonic transformations* of post-Newtonian theory. We describe a simple application of this formalism in Sec. 8.3.5, in which we examine the Newtonian potential of a moving body in its own (non-inertial) reference frame.

The post-Newtonian and harmonic transformations typically produce gravitational potentials that contain spatially-growing terms, even when the original potentials vanish in the formal limit $r \to \infty$. In Sec. 8.3.6 we specialize them further by demanding that the transformed potentials continue to vanish in the limit $\bar{r} \to \infty$; this property defines what is known as the *post-Galilean transformations* of post-Newtonian theory.

Within the post-Newtonian class of transformations there exists an interesting subclass that corresponds very closely to the ordinary gauge transformations of electrodynamics. We examine these in Sec. 8.3.7, and introduce the so-called *standard gauge* of post-Newtonian theory. This was the gauge that was adopted by Chandrasekhar in his pioneering work on the subject, and much of the older post-Newtonian literature is framed in this gauge. The standard gauge, however, has become less popular of late, and the more recent literature is uniformly cast in the harmonic gauge. We adhere to this choice in most of the book, but the standard post-Newtonian gauge is featured in Chapter 13, in which we examine alternative theories of gravity.

In this section we follow closely the treatment of post-Newtonian coordinate transformations developed by our friends Étienne Racine and Éanna Flanagan (2005). We recall that under the transformation of Eq. (8.33), the components of the metric tensor change according to

$$\bar{g}_{00} = \left(\frac{\partial t}{\partial \bar{t}}\right)^2 g_{00} + \frac{2}{c}\frac{\partial t}{\partial \bar{t}}\frac{\partial x^j}{\partial \bar{t}} g_{0j} + \frac{1}{c^2}\frac{\partial x^j}{\partial \bar{t}}\frac{\partial x^k}{\partial \bar{t}} g_{jk}, \tag{8.34a}$$

$$\bar{g}_{0j} = c\frac{\partial t}{\partial \bar{t}}\frac{\partial t}{\partial \bar{x}^j} g_{00} + \left(\frac{\partial t}{\partial \bar{t}}\frac{\partial x^k}{\partial \bar{x}^j} + \frac{\partial x^k}{\partial \bar{t}}\frac{\partial t}{\partial \bar{x}^j}\right) g_{0k} + \frac{1}{c}\frac{\partial x^k}{\partial \bar{t}}\frac{\partial x^n}{\partial \bar{x}^j} g_{kn}, \tag{8.34b}$$

$$\bar{g}_{jk} = c^2\frac{\partial t}{\partial \bar{x}^j}\frac{\partial t}{\partial \bar{x}^k} g_{00} + c\left(\frac{\partial t}{\partial \bar{x}^j}\frac{\partial x^n}{\partial \bar{x}^k} + \frac{\partial x^n}{\partial \bar{x}^j}\frac{\partial t}{\partial \bar{x}^k}\right) g_{0n} + \frac{\partial x^n}{\partial \bar{x}^j}\frac{\partial x^p}{\partial \bar{x}^k} g_{np}. \tag{8.34c}$$

Our most important results are summarized in Box 8.2. The reader is invited to peruse the summary before getting started with the details, so as to benefit from an overview of what is to come.

Box 8.2 **Post-Newtonian transformations**

The most general coordinate transformation that preserves the post-Newtonian ordering of the metric is given by

$$t = \bar{t} + \frac{1}{c^2}\alpha(\bar{t}, \bar{x}^j) + \frac{1}{c^4}\beta(\bar{t}, \bar{x}^j) + O(c^{-6}),$$
$$x^j = \bar{x}^j + r^j(\bar{t}) + \frac{1}{c^2}h^j(\bar{t}, \bar{x}^j) + O(c^{-4}),$$

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where

$$\alpha = A(\bar{t}) + v_j \bar{x}^j,$$
$$h^j = H^j(\bar{t}) + H^j_k(\bar{t})\bar{x}^k + \frac{1}{2}H^j_{kn}(\bar{t})\bar{x}^k\bar{x}^n,$$

with

$$H_{jk} = \epsilon_{jkn}R^n(\bar{t}) + \frac{1}{2}v_j v_k - \delta_{jk}\Big(\dot{A} - \frac{1}{2}v^2\Big),$$
$$H_{jkn} = -\delta_{jk}a_n - \delta_{jn}a_k + \delta_{kn}a_j.$$

The functions A, r^j, H^j, and R^j are freely specifiable functions of time $\bar{t}$, while β is a free function of all the coordinates. The transformation is therefore characterized by ten arbitrary functions of time, and one arbitrary function of all the coordinates. An overdot indicates differentiation with respect to $\bar{t}$, and we have introduced $v^j := \dot{r}^j$ and $a^j := \dot{v}^j = \ddot{r}^j$. In addition, we let $v^2 = \delta_{jk}v^j v^k$.

The transformation preserves the post-Newtonian ordering of the metric, but it does not necessarily keep the coordinates harmonic. To preserve this also we must set

$$\beta = \frac{1}{6}\ddot{A}\delta_{jk}\bar{x}^j\bar{x}^k + \frac{1}{30}\big(\delta_{jk}\dot{a}_n + \delta_{jn}\dot{a}_k + \delta_{kn}\dot{a}_j\big)\bar{x}^j\bar{x}^k\bar{x}^n + \gamma(\bar{t}, \bar{x}^j),$$

and γ is required to satisfy Laplace's equation: $\bar{\nabla}^2\gamma = 0$, with $\bar{\nabla}^2$ denoting the Laplacian operator in the coordinates $\bar{x}^j$. The arbitrary function β has therefore been replaced by an arbitrary harmonic function γ.

Under a harmonic coordinate transformation the potentials become

$$\bar{U}(\bar{t}, \bar{x}^j) = \hat{U} - \dot{A} + \frac{1}{2}v^2 - a_j\bar{x}^j,$$
$$\bar{U}^j(\bar{t}, \bar{x}^j) = \hat{U}^j - v^j\hat{U} + \frac{1}{4}\bigg(V^j + V^j_k\bar{x}^k + \frac{1}{2}V^j_{kn}\bar{x}^k\bar{x}^n + \partial_{\bar{j}}\gamma\bigg),$$
$$\begin{aligned}\bar{\Psi}(\bar{t}, \bar{x}^j) = \hat{\Psi} &- 4v^j\hat{U}_j + 2v^2\hat{U} + \big(A + v_k\bar{x}^k\big)\partial_{\bar{t}}\hat{U} \\ &+ \Big(F^j + F^j_k\bar{x}^k + \frac{1}{2}F^j_{kn}\bar{x}^k\bar{x}^n\Big)\partial_{\bar{j}}\hat{U} \\ &+ G + G_j\bar{x}^j + \frac{1}{2}G_{jk}\bar{x}^j\bar{x}^k + \frac{1}{6}G_{jkn}\bar{x}^j\bar{x}^k\bar{x}^n - \partial_{\bar{t}}\gamma,\end{aligned}$$

where

$$V^j = (2\dot{A} - v^2)v^j - \dot{H}^j + \epsilon^j_{\ pq}v^p R^q,$$
$$V^j_k = \frac{3}{2}v^j a_k + \frac{1}{2}a^j v_k + \delta^j_k\Big(\frac{4}{3}\ddot{A} - 2v^n a_n\Big) - \epsilon^j_{\ kp}\dot{R}^p,$$
$$V^j_{kn} = \frac{6}{5}\big(\delta^j_k\dot{a}_n + \delta^j_n\dot{a}_k\big) - \frac{4}{5}\delta_{kn}\dot{a}^j,$$

(*continued overleaf*)

$$
\begin{aligned}
F^j &= H^j - A v^j, \\
F^j_{\ k} &= -\delta^j_k \Bigl(\dot{A} - \frac{1}{2}v^2\Bigr) - \frac{1}{2} v^j v_k + \epsilon^j_{\ kp} R^p, \\
F^j_{\ kn} &= -\bigl(\delta^j_k a_n + \delta^j_n a_k\bigr) + \delta_{kn} a^j,
\end{aligned}
$$

$$
\begin{aligned}
G &= \frac{1}{2}\dot{A}^2 - \dot{A} v^2 + \frac{1}{4} v^4 + \dot{H}^j v_j, \\
G_j &= \Bigl(\dot{A} - \frac{1}{2}v^2\Bigr) a_j - \Bigl(\ddot{A} - \frac{3}{2} v^k a_k\Bigr) v_j - \epsilon_{jpq} v^p \dot{R}^q, \\
G_{jk} &= a_j a_k - v_j \dot{a}_k - \dot{a}_j v_k + \delta_{jk}(v_n \dot{a}^n) - \frac{1}{3}\delta_{jk}\ddot{A}, \\
G_{jkn} &= -\frac{1}{5}\bigl(\delta_{jk}\ddot{a}_n + \delta_{jn}\ddot{a}_k + \delta_{kn}\ddot{a}_j\bigr).
\end{aligned}
$$

The "hatted" potentials are equal to the original potentials evaluated at time $t = \bar{t}$ and position $x^j = \bar{x}^j + r^j(\bar{t})$. For example,

$$
\hat{U}(\bar{t}, \bar{x}^j) := U\bigl(t = \bar{t}, x^j = \bar{x}^j + r^j(\bar{t})\bigr).
$$

Because U now possesses, in addition to its original explicit time dependence, an implicit time dependence contained in $r^j(\bar{t})$, some care must be exercised when taking time derivatives. We have

$$
\frac{\partial \hat{U}}{\partial \bar{t}} = \frac{\partial U}{\partial t} + v^j \frac{\partial U}{\partial x^j}, \qquad \frac{\partial \hat{U}}{\partial \bar{x}^j} = \frac{\partial U}{\partial x^j},
$$

in which the right-hand-sides are evaluated at $t = \bar{t}$ and $x^j = \bar{x}^j + r^j(\bar{t})$.

8.3.2 Newtonian transformations

We begin with a search for a coordinate transformation that preserves the form of the metric at Newtonian order:

$$
g_{00} = -1 + \frac{2}{c^2} U + O(c^{-4}), \qquad g_{0j} = O(c^{-3}), \qquad g_{jk} = \delta_{jk} + O(c^{-2}). \tag{8.35}
$$

Specifically, we demand that the new metric takes the form

$$
\bar{g}_{00} = -1 + \frac{2}{c^2} \bar{U} + O(c^{-4}), \qquad \bar{g}_{0j} = O(c^{-3}), \qquad \bar{g}_{jk} = \delta_{jk} + O(c^{-2}), \tag{8.36}
$$

with a new Newtonian potential $\bar{U}$ whose relation with the old one will be determined by the transformation.

Inspecting Eq. (8.34a) first, we note that the leading term in $\bar{g}_{00}$ will have the correct value of -1 if and only if $\partial t/\partial \bar{t} = 1 + O(c^{-2})$. Moving next to Eq. (8.34c), we see that $\bar{g}_{jk}$ will contain unwanted terms of order c^2 unless $\partial t/\partial \bar{x}^j = O(c^{-2})$. And the terms of order

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c^0 will have the correct form if and only if

$$\delta_{jk} = \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\delta_{pq} + O(c^{-2}). \tag{8.37}$$

The Newtonian transformation must preserve the form of the spatial metric, and this means that it must be the combination of a translation of the spatial origin with a rotation of the coordinate axes: $x^j = r^j + R^j_{\ k}\bar{x}^k + O(c^{-2})$. Here $r^j(\bar{t})$ are arbitrary functions of time, and $R^j_{\ k}(\bar{t})$ are the components of a rotation matrix that satisfies $\delta_{pq}R^p_{\ j}R^q_{\ k} = \delta_{jk}$. At this stage of our considerations we conclude that the transformation must take the form

$$t = \bar{t} + \frac{1}{c^2}\alpha(\bar{t}, \bar{x}^j) + O(c^{-4}), \qquad x^j = r^j(\bar{t}) + R^j_{\ k}(\bar{t})\bar{x}^k + O(c^{-2}), \tag{8.38}$$

where α is (for now) an arbitrary function of the new coordinates, and r^j, $R^j_{\ k}$ are arbitrary functions of time.

We next examine Eq. (8.34b), in which we make the substitutions of Eq. (8.38). After simplification we notice that $\bar{g}_{0j}$ will contain unwanted terms at order c^{-1} unless

$$\partial_{\bar{j}}\alpha = \left(v^k + \dot{R}^k_{\ n}\bar{x}^n\right)R_{kj}; \tag{8.39}$$

an overdot indicates differentiation with respect to $\bar{t}$, and we introduced the notation $v^j := \dot{r}^j$. The previously stated condition on the rotation matrix implies that $\dot{R}^k_n R_{kj}$ is antisymmetric on n and j, and can therefore be expressed as $\dot{R}^k_n R_{kj} = \epsilon_{njm}\omega^m(t)$ for some vector ω^m. The equation for α is now

$$\partial_{\bar{j}}\alpha = v^k R_{kj} + \epsilon_{jmn}\omega^m\bar{x}^n\,, \tag{8.40}$$

and we see that the final term cannot be written as the gradient of any function. To eliminate the unwanted c^{-1} term in $\bar{g}_{0j}$ we must therefore set $\omega^m = 0$, so that $R^j_{\ k}$ describes a time-independent rotation of the coordinate axes. We choose to discard this uninteresting coordinate freedom by setting

$$R_{jk} = \delta_{jk}. \tag{8.41}$$

The general solution for α is then

$$\alpha = A(\bar{t}) + v_j(\bar{t})\bar{x}^j, \tag{8.42}$$

in which A is an arbitrary function of time $\bar{t}$. The Newtonian transformation of Eq. (8.38) is now fully characterized.

Box 8.3 **Rotating coordinates**

By forcing $\bar{g}_{0j}$ to vanish at order c^{-1}, we are consciously excluding rotating coordinate systems from the allowed class of post-Newtonian coordinates. To describe a purely rotating coordinate system, we would use the transformation

$$t = \bar{t} + O(c^{-4}), \qquad x^j = R^j_{\ k}(\bar{t})\bar{x}^k + O(c^{-2})\,,$$

which leads to a metric of the form

$$\bar{g}_{00} = -1 + \frac{2}{c^2}\bar{U} + \frac{1}{c^2}\big[\omega^2 - (\boldsymbol{\omega}\cdot\bar{\boldsymbol{n}})^2\big]\bar{r}^2 + O(c^{-4}),$$
$$\bar{g}_{0j} = \frac{1}{c}(\boldsymbol{\omega}\times\bar{\boldsymbol{x}})_j + O(c^{-3}),$$
$$\bar{g}_{jk} = \delta_{jk} + O(c^{-2}),$$

where $\omega_j(t) := \frac{1}{2}\epsilon_{jlm}\dot{R}^k_{\ l}R_{km}$ and $\bar{\boldsymbol{n}} = \bar{\boldsymbol{x}}/\bar{r}$. From a relativistic point of view, this metric presents a number of problems. One of these is that it does not reduce to the Minkowski metric when $\bar{r} \to \infty$. An even worse problem is that g_{00} vanishes when $\omega\bar{r}\sin\theta = c(1 - U/c^2)$, where θ is the angle between $\boldsymbol{\omega}$ and $\bar{\boldsymbol{n}}$. This "light cylinder" is a place where the speed of a particle at rest in the rotating frame equals the local speed of light as measured in the global, non-rotating frame; particles at rest outside the light cylinder exceed the local speed of light.

Other issues connected with the rotating frame include the inability to synchronize clocks consistently around a circle at rest in this frame (the Sagnac effect, reviewed in Sec. 10.3.4), and the common misconception that the circumference of a rotating disk is shortened compared to 2π times its radius. These issues have generated so much misunderstanding that in the general relativity textbook by H.P. Robertson and T.W. Noonan (1968), there is a paragraph on this topic headed "That darned* rotating disk," with the asterisk indicating that the actual word selected by Robertson in his original lecture notes was much stronger!

As indicated in the text, we wish to preserve the post-Newtonian expansion of the metric, and therefore exclude rotating coordinate systems from our considerations. This doesn't mean, however, that rotating coordinates are never appropriate. They can indeed be very useful, provided that one stays well within the light cylinder. For example, a coordinate system that rotates with the Earth is an extremely powerful tool to describe post-Newtonian gravity in and around the Earth, including its effects on geocentric satellites, atomic timekeeping, and the Global Positioning System. These applications are discussed in detail in Chapter 10.

To determine how the Newtonian potential changes under the transformation, we return to Eq. (8.34a) and examine the terms of order c^{-2} after making the substitutions of Eq. (8.38). After simplification we find that the right-hand side is given by $-1 + 2c^{-2}(U - \partial_{\bar{t}}\alpha + \frac{1}{2}v^2) + O(c^{-4})$, where $v^2 := \delta_{jk}v^j v^k$. The new potential must therefore be $\bar{U} = U - \partial_{\bar{t}}\alpha + \frac{1}{2}v^2$. Here $\bar{U}$ is expressed in terms of the new coordinates $(\bar{t}, \bar{x}^j)$, but U is still written in terms of the old coordinates (t, x^j). To make the equation more useful we should express U as a function of the new coordinates. To achieve this we write

$$U(t, x^j) = U(\bar{t} + c^{-2}\alpha + \cdots, \bar{x}^j + r^j + \cdots) \tag{8.43}$$

and perform a Taylor expansion of the right-hand side about the point $(\bar{t}, \bar{x}^j + r^j)$. This gives $U(t, x^j) = U(\bar{t}, \bar{x}^j + r^j) + O(c^{-2})$, and we find that the terms of order c^{-2} play no role in the transformation of the Newtonian potential. (They do, however, appear in the post-Newtonian transformation of the following subsection.)

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To distinguish clearly between the sets of arguments $(\bar{t}, \bar{x}^j)$, $(\bar{t}, \bar{x}^j + r^j)$, and (t, x^j) we introduce the "hatted" potential

$$\hat{U}(\bar{t}, \bar{x}^j) := U(\bar{t}, \bar{x}^j + r^j). \tag{8.44}$$

This is the original potential U evaluated at time $t = \bar{t}$ and position $x^j = \bar{x}^j + r^j(\bar{t})$. In terms of this we have

$$U(t, x^j) = \hat{U}(\bar{t}, \bar{x}^j) + O(c^{-2}) \tag{8.45}$$

and the transformed potential is $\bar{U} = \hat{U} - \partial_{\bar{t}}\alpha + \frac{1}{2}v^2$. Using Eq. (8.42), this is

$$\bar{U} = \hat{U} - \dot{A} + \frac{1}{2}v^2 - a_j \bar{x}^j, \tag{8.46}$$

where $a^j := \dot{v}^j = \ddot{r}^j$. All members of this equation are functions of the new coordinates $(\bar{t}, \bar{x}^j)$.

When it is expressed in terms of U as in Eq. (8.44), the hatted potential $\hat{U}$ possesses both an explicit and an implicit dependence upon the time coordinate $\bar{t}$. The explicit dependence is contained in U's temporal argument, while the implicit dependence appears via $r^j(\bar{t})$ in the spatial arguments. Some care must therefore be exercised when computing partial derivatives. We have, for example,

$$\frac{\partial \hat{U}}{\partial \bar{t}} = \left(\frac{\partial U}{\partial t} + v^j \frac{\partial U}{\partial x^j} \right)_{t=\bar{t},\ x=\bar{x}+r}, \tag{8.47}$$

in which the substitutions $t = \bar{t}$, $x^j = \bar{x}^j + r^j(\bar{t})$ are made after differentiating U with respect to its original variables t and x^j. Spatial derivatives, on the other hand, are given simply by

$$\frac{\partial \hat{U}}{\partial \bar{x}^j} = \left(\frac{\partial U}{\partial x^j} \right)_{t=\bar{t},\ x=\bar{x}+r}. \tag{8.48}$$

8.3.3 Post-Newtonian transformations

To proceed to the next order we write the coordinate transformation as

$$t = \bar{t} + \frac{1}{c^2}\alpha(\bar{t}, \bar{x}^j) + \frac{1}{c^4}\beta(\bar{t}, \bar{x}^j) + O(c^{-6}), \tag{8.49a}$$

$$x^j = \bar{x}^j + r^j(\bar{t}) + \frac{1}{c^2}h^j(\bar{t}, \bar{x}^j) + O(c^{-4}), \tag{8.49b}$$

where α is given by Eq. (8.42), while β and h^j represent the additional coordinate freedom that appears at 1PN order.

The function β will remain arbitrary. To constrain h^j we examine the $O(c^{-2})$ terms on the right-hand-side of Eq. (8.34c) and demand that, in accordance with Eq. (8.20), they be equal to $2\bar{U}\delta_{jk}$. With Eqs. (8.45) and (8.46), we find after simplification that h^j must be a

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solution to the differential equation

$$\partial_{\bar{j}} h_k + \partial_{\bar{k}} h_j = -2\delta_{jk}\left(\dot{A} - \frac{1}{2}v^2 + a_n \bar{x}^n\right) + v_j v_k. \tag{8.50}$$

The general solution to this equation is the sum of a particular solution and the general solution to the homogeneous equation $\partial_{\bar{j}} h_k + \partial_{\bar{k}} h_j = 0$. (The expert will recognize this as Killing's equation in a flat, three-dimensional space.)

The form of the homogeneous equation reveals that $h^j_{\rm hom}$ must be linear in the coordinates: $h^j_{\rm hom} = H^j + A^j{}_k \bar{x}^k$, where H^j and $A^j{}_k$ are functions of $\bar{t}$. Substitution within the differential equation reveals that $H^j(\bar{t})$ is arbitrary, but that A_{jk} must be an antisymmetric tensor. Such a tensor contains three independent components, and we can always express it as $A_{jk} = \epsilon_{jkn} R^n$, in terms of a vector R^n that also contains three independent components. We have obtained $h^j_{\rm hom} = H^j(\bar{t}) + \epsilon^j{}_{kn} \bar{x}^k R^n(\bar{t})$. The first term represents a translational component to the coordinate transformation, which combines with the Newtonian translation to form the total translation $r^j + c^{-2} H^j$. The second term represents a rotation of the coordinate axes, and the rotation matrix is $R^j{}_k = c^{-2} \epsilon^j{}_{kn} R^n$.

The form of the inhomogeneous equation reveals that a particular solution $h^j_{\rm part}$ will be quadratic in the coordinates. We adopt $B_{jk}\bar{x}^k + \frac{1}{2} B_{jkn} \bar{x}^k \bar{x}^n$ as a trial solution, and observe that B_{jkn} can be chosen to be symmetric in the last pair of indices. Substitution within the differential equation shows that B_{jk} is constrained by $B_{jk} + B_{kj} = -2\delta_{jk}(\dot{A} - \frac{1}{2}v^2) + v_j v_k$, while B_{jkn} is constrained by $B_{jkn} + B_{kjn} = -2\delta_{jk} a_n$. The solutions are readily identified as $B_{jk} = -\delta_{jk}(\dot{A} - \frac{1}{2}v^2) + \frac{1}{2} v_j v_k$ and $B_{jkn} = -\delta_{jk} a_n - \delta_{jn} a_k + \delta_{kn} a_j$.

Collecting results, we write our final expression for $h^j = h^j_{\rm hom} + h^j_{\rm part}$ as

$$h^j = H^j(\bar{t}) + H^j{}_k(\bar{t})\bar{x}^k + \frac{1}{2} H^j{}_{kn}(\bar{t}) \bar{x}^k \bar{x}^n, \tag{8.51}$$

with

$$H_{jk} = \epsilon_{jkn} R^n(\bar{t}) + \frac{1}{2} v_j v_k - \delta_{jk}\left(\dot{A} - \frac{1}{2}v^2\right), \tag{8.52a}$$

$$H_{jkn} = -\delta_{jk} a_n - \delta_{jn} a_k + \delta_{kn} a_j. \tag{8.52b}$$

This piece of the coordinate transformation involves the six arbitrary functions of time that are contained in $H^j(\bar{t})$ and $R^j(\bar{t})$.

To determine how the vector potential U^j transforms under the post-Newtonian transformation of Eqs. (8.49), we make the substitutions in Eq. (8.34) and demand that g_{0j} keeps its post-Newtonian form of $-4c^{-3} U_j + O(c^{-5})$. A careful evaluation of Eq. (8.34b) reveals that the new vector potential $\bar{U}_j$ is given by

$$4\bar{U}_j = 4(\hat{U}_j - v_j \hat{U}) + \partial_{\bar{j}} \beta + v_j \partial_{\bar{t}} \alpha - v^k \partial_{\bar{j}} h_k - \partial_{\bar{t}} h_j, \tag{8.53}$$

in which $\hat{U}_j$ is defined by analogy with Eq. (8.44). Taking into account Eqs. (8.42) and (8.51), we finally arrive at

$$\bar{U}_j = \hat{U}_j - v_j \hat{U} + \frac{1}{4}\left(V_j + V_{jk}\bar{x}^k + \frac{1}{2} V_{jkn} \bar{x}^k \bar{x}^n + \partial_{\bar{j}} \beta\right), \tag{8.54}$$

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with

$$V_j = (2\dot{A} - v^2)v_j - \dot{H}_j + \epsilon_{jpq} v^p R^q, \tag{8.55a}$$

$$V_{jk} = \frac{3}{2} v_j a_k + \frac{1}{2} v_k a_j + \delta_{jk}\big(\ddot{A} - 2v^n a_n\big) - \epsilon_{jkp} \dot{R}^p, \tag{8.55b}$$

$$V_{jkn} = \delta_{jk}\dot{a}_n + \delta_{jn}\dot{a}_k - \delta_{kn}\dot{a}_j. \tag{8.55c}$$

We are now ready to derive the transformation equation for the post-Newtonian potential Ψ. We proceed along the same lines as for the vector potential, but before we begin we must recover the $O(c^{-2})$ terms that were discarded back in Eq. (8.45); these were not needed previously, but they appear in the transformed version of g_{00} at order c^{-4}. If we write $U(t, x^j)$ as $U(\bar{t} + c^{-2}\alpha + \cdots, \bar{x}^j + r^j + c^{-2}h^j + \cdots)$ and expand to first order in c^{-2}, we obtain

$$U(t, x^j) = \hat{U} + \frac{1}{c^2}\alpha\Big(\partial_{\bar{t}}\hat{U} - v^j\partial_{\bar{j}}\hat{U}\Big) + \frac{1}{c^2}h^j\partial_{\bar{j}}\hat{U} + O(c^{-4}); \tag{8.56}$$

the right-hand side is expressed as a function of $(\bar{t}, \bar{x}^j)$, and we have used Eqs. (8.47) and (8.48) to relate the partial derivatives of U to those of $\hat{U}$. After substitution of Eq. (8.49) into Eq. (8.34a) we find that the new post-Newtonian potential must be given by

$$\begin{aligned}\bar{\Psi} &= \hat{\Psi} + 2v^2\hat{U} + \alpha\partial_{\bar{t}}\hat{U} + (h^j - \alpha v^j)\partial_{\bar{j}}\hat{U} - 4v_j\hat{U}^j \\ &\quad + \frac{1}{2}(\partial_{\bar{t}}\alpha)^2 - v^2\partial_{\bar{t}}\alpha + \frac{1}{4}v^4 - \partial_{\bar{t}}\beta + v_j\partial_{\bar{t}}h^j.\end{aligned} \tag{8.57}$$

After taking into account Eqs. (8.42) and (8.51), we finally arrive at

$$\begin{aligned}\bar{\Psi} &= \hat{\Psi} - 4v_j\hat{U}^j + 2v^2\hat{U} + \big(A + v_k\bar{x}^k\big)\partial_{\bar{t}}\hat{U} + \Big(F^j + F^j_{\ k}\bar{x}^k + \frac{1}{2}F^j_{\ kn}\bar{x}^k\bar{x}^n\Big)\partial_{\bar{j}}\hat{U} \\ &\quad + G + G_j\bar{x}^j + \frac{1}{2}G_{jk}\bar{x}^j\bar{x}^k - \partial_{\bar{t}}\beta,\end{aligned} \tag{8.58}$$

with

$$F^j = H^j - Av^j, \tag{8.59a}$$

$$F^j_{\ k} = -\delta^j_{\ k}\Big(\dot{A} - \frac{1}{2}v^2\Big) - \frac{1}{2}v^j v_k + \epsilon^j_{\ kp}R^p, \tag{8.59b}$$

$$F^j_{\ kn} = -\big(\delta^j_{\ k}a_n + \delta^j_{\ n}a_k\big) + \delta_{kn}a^j, \tag{8.59c}$$

$$G = \frac{1}{2}\dot{A}^2 - \dot{A}v^2 + \frac{1}{4}v^4 + \dot{H}^j v_j, \tag{8.59d}$$

$$G_j = \Big(\dot{A} - \frac{1}{2}v^2\Big)a_j - \Big(\ddot{A} - \frac{3}{2}v^k a_k\Big)v_j - \epsilon_{jpq}v^p\dot{R}^q, \tag{8.59e}$$

$$G_{jk} = a_j a_k - v_j\dot{a}_k - \dot{a}_j v_k + \delta_{jk}(v_n\dot{a}^n). \tag{8.59f}$$

The hatted potential $\hat{\Psi}$ is defined by analogy with Eq. (8.44).

Our task of constructing the most general coordinate transformation that preserves the post-Newtonian form of the metric is now complete. At Newtonian order the transformation is characterized by an arbitrary translation $r^j(\bar{t})$ and a shift $\alpha = A(\bar{t}) + v_j(\bar{t})\bar{x}^j$ of the time coordinate at order c^{-2}. At post-Newtonian order the transformation involves an additional component $H^j(\bar{t})$ to the translation, as well as a rotation governed by the vector $R^j(\bar{t})$. In addition, the transformation involves an arbitrary shift $\beta(\bar{t}, \bar{x}^j)$ of the time coordinate at order c^{-4}. All in all we have ten arbitrary functions of time, and one free function β of all the coordinates. The transformed potentials $\bar{U}$, $\bar{U}^j$, and $\bar{\Psi}$ are obtained from the old ones by employing Eqs. (8.46), (8.54), and (8.58), respectively.

8.3.4 Harmonic transformations

The general transformation of the preceding subsection does not, in general, preserve the harmonic condition of Eq. (8.21). It is possible, however, to specialize $\beta(\bar{t}, \bar{x}^j)$ so that we also have

$$\partial_{\bar{t}}\bar{U} + \partial_{\bar{j}}\bar{U}^j = 0 \tag{8.60}$$

in the new coordinates. This restriction of the coordinate freedom defines what we shall call the class of *harmonic coordinate transformations*.

In view of Eq. (8.47) we find that the harmonic condition is

$$\partial_{\bar{t}}\hat{U} - v^j\partial_{\bar{j}}\hat{U} + \partial_{\bar{j}}\hat{U}^j = 0 \tag{8.61}$$

when it is expressed in terms of the hatted potentials. If we substitute Eqs. (8.46) and (8.54) into Eq. (8.60) and make use of Eq. (8.61), we find that the harmonic condition is preserved when β satisfies the Poisson equation

$$\bar{\nabla}^2\beta = \ddot{A} + \dot{a}_j\bar{x}^j. \tag{8.62}$$

Here $\bar{\nabla}^2$ is the Laplacian operator in the coordinates $\bar{x}^j$. The general solution to this equation is the sum of a particular solution and the general solution to Laplace's equation. The particular solution must be cubic in the coordinates, and we construct it with the help of the ansatz $\frac{1}{2}C_{jk}\bar{x}^j\bar{x}^k + \frac{1}{6}C_{jkn}\bar{x}^j\bar{x}^k\bar{x}^n$, in which C_{jk} and C_{jkm} depend on $\bar{t}$ and are completely symmetric tensors. This property and the differential equation imply that $C_{jk} = \frac{1}{3}\delta_{jk}\ddot{A}$ and $C_{jkn} = \frac{1}{5}(\delta_{jk}\dot{a}_n + \delta_{jn}\dot{a}_k + \delta_{kn}\dot{a}_j)$. We have obtained

$$\beta = \frac{1}{6}\ddot{A}\delta_{jk}\bar{x}^j\bar{x}^k + \frac{1}{30}\big(\delta_{jk}\dot{a}_n + \delta_{jn}\dot{a}_k + \delta_{kn}\dot{a}_j\big)\bar{x}^j\bar{x}^k\bar{x}^n + \gamma(\bar{t}, \bar{x}^j), \tag{8.63}$$

where γ is any harmonic function that satisfies $\bar{\nabla}^2\gamma = 0$.

Making the substitution in Eqs. (8.46), (8.54), and (8.58), we obtain the results listed in Box 8.2. [Note that the expressions for V_{jk}, V_{jkn}, G_{jk}, and G_{jkn} that appear in the Box are different from those given by Eqs. (8.55) and (8.59). The differences are accounted for by the terms generated by $\partial_{\bar{j}}\beta$ and $\partial_{\bar{t}}\beta$.] The transformation is still characterized by the ten arbitrary functions of time that are contained in $A(\bar{t})$, $r^j(\bar{t})$, $H^j(\bar{t})$, and $R^j(\bar{t})$, but the remaining freedom is now restricted to a harmonic function $\gamma(\bar{t}, \bar{x}^j)$.

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8.3.5 Comoving frame of a moving body

The general post-Newtonian transformations, and the restricted class of harmonic transformations, contain an enormous amount of freedom, and the transformations introduce spatially-growing terms in the potentials. For example, the transformation of the Newtonian potential is

$$\bar{U} = \hat{U} - \dot{A} + \frac{1}{2}v^2 - a_j \bar{x}^j, \tag{8.64}$$

and the last term grows linearly with $\bar{r}$. Similarly, $\bar{U}^j$ contains terms that grow like $\bar{r}^2$, and in the harmonic case, $\bar{\Psi}$ grows like $\bar{r}^3$. In view of this situation, a natural question to ponder is: what purpose is there in all this coordinate freedom?

We shall have occasion to give a more complete answer to this question in Sec. 9.4, but here we consider a simple application of the formalism that should illustrate its usefulness. We consider a spherical body of mass m whose center-of-mass it situated at $x^j = r^j(t)$ in an inertial frame of reference. The body creates a gravitational potential $U_{\rm body}$, and it is surrounded by an external matter distribution that creates a potential $U_{\rm ext}$. The total potential is $U = U_{\rm body} + U_{\rm ext}$, and we wish to examine its form in the non-inertial frame attached to the moving body. The coordinate transformation is given by $t = \bar{t} + c^{-2}\alpha + O(c^{-4})$ and $x^j = \bar{x}^j + r^j(\bar{t}) + O(c^{-2})$, with $\alpha = A(\bar{t}) + v_j \bar{x}^j$.

In the original (inertial) coordinates we have that the potential outside the body is given by

$$U_{\rm body} = \frac{Gm}{|\boldsymbol{x} - \boldsymbol{r}(t)|}. \tag{8.65}$$

To simplify its expression we expand the external potential in a Taylor series about $x^j = r^j(t)$:

$$U_{\rm ext} = U(t, r^j) + (x-r)^j \partial_j U_{\rm ext}(t, r^j) + \frac{1}{2}(x-r)^j (x-r)^k \partial_{jk} U_{\rm ext}(t, r^j) + \cdots \tag{8.66}$$

The hatted potentials are

$$\hat{U}_{\rm body} = \frac{Gm}{\bar{r}} \tag{8.67}$$

and

$$\hat{U}_{\rm ext} = U_{\rm ext}(\bar{t}, r^j) + \bar{x}^j \partial_j U_{\rm ext}(\bar{t}, r^j) + \frac{1}{2}\bar{x}^j \bar{x}^k \partial_{jk} U_{\rm ext}(\bar{t}, r^j) + \cdots \tag{8.68}$$

We see that the total potential $\hat{U} = \hat{U}_{\rm body} + \hat{U}_{\rm ext}$ naturally contains growing terms that are associated with the external matter, in addition to the decaying term that is associated with the reference body. Note that after each differentiation, the external potential is evaluated at $t = \bar{t}$ and $x^j = r^j(\bar{t})$.

The transformed potential in the comoving frame of the body is $\bar{U} = \bar{U}_{\rm body} + \bar{U}_{\rm ext}$, with

$$\bar{U}_{\rm body} = \frac{Gm}{\bar{r}} \tag{8.69}$$

and

$$\bar{U}_{\rm ext} = \left[U_{\rm ext}(\bar{t}, r^j) - \dot{A} + \frac{1}{2}v^2 \right] + \bar{x}^j \Big[a_j - \partial_j U_{\rm ext}(\bar{t}, r^j) \Big] + \frac{1}{2}\bar{x}^j \bar{x}^k \partial_{jk} U_{\rm ext}(\bar{t}, r^j) + \cdots \tag{8.70}$$

We simplify this by first exploiting the coordinate freedom, which allows us to set

$$\dot{A} = \frac{1}{2}v^2 + U_{\rm ext}(\bar{t}, r^j); \tag{8.71}$$

this is a differential equation that determines $A(\bar{t})$ up to an uninteresting constant of integration. We also make use of the fact that our body moves according to the Newtonian equations of motion, so that

$$a_j = \partial_j U_{\rm ext}(\bar{t}, r^j). \tag{8.72}$$

We recall that a^j stands for $d^2 r^j / d\bar{t}^2$.

Our end result for the comoving-frame gravitational potential is

$$\bar{U} = \frac{Gm}{\bar{r}} + \bar{U}_{\rm tidal}, \tag{8.73}$$

where

$$\bar{U}_{\rm tidal} = \frac{1}{2}\bar{x}^j \bar{x}^k \partial_{jk} U_{\rm ext}(\bar{t}, r^j) + \cdots \tag{8.74}$$

is what remains of the external potential. As its label indicates, it is this potential that is responsible for the tidal interaction between the moving body and the external matter distribution. The Newtonian physics of tidally deformed bodies was explored in some detail in Sec. 2.5.

The coordinate transformation that takes us from the inertial frame to the moving frame is

$$t = \bar{t} + \frac{1}{c^2} \int \left[\frac{1}{2}v^2 + U_{\rm ext}(\bar{t}, r^j) \right] d\bar{t} + \frac{1}{c^2} v_j(\bar{t}) \bar{x}^j + O(c^{-4}) \tag{8.75}$$

and

$$x^j = \bar{x}^j + r^j(\bar{t}) + O(c^{-2}). \tag{8.76}$$

The transformation can of course be generalized to post-Newtonian order, and we go through this exercise in Sec. 9.4.

8.3.6 Post-Galilean transformations

As we have seen, it can prove useful to exploit the full freedom contained in the general post-Newtonian transformations, or the restricted class of harmonic transformations, when one considers a bounded domain of spacetime such as the neighborhood of a moving body. When the considerations are global, however, the general freedom is too vast, and one would like to constrain it so as to eliminate the spatially-growing terms in the potentials. In this section we assume that the original potentials U, U^j, and Ψ vanish in the formal limit

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$r \to \infty$, and we specialize the post-Newtonian transformations so that the new potentials $\bar{U}$, $\bar{U}^j$, and $\bar{\Psi}$ share this property. This restricted class of coordinate transformations is known as the *post-Galilean transformations* of post-Newtonian theory. The name was coined by Chandrasekhar and Contopoulos in their classic 1967 paper.

Construction

Inspection of Eq. (8.46) reveals that the Newtonian potential will grow linearly with $\bar{r}$ unless $a^j = 0$. Discarding an uninteresting constant translation of the coordinates, this means that $r^j(\bar{t})$ must be of the form

$$r^j = V^j \bar{t}, \tag{8.77}$$

with V^j a constant vector. To eliminate the spatially-constant term in $\bar{U}$ we must also set $\dot{A} = \frac{1}{2}V^2$, so that

$$A = \frac{1}{2}V^2\bar{t}. \tag{8.78}$$

Here $V^2 := \delta_{jk}V^jV^k$, and we again discard an uninteresting integration constant. These results imply that the post-Galilean transformation leaves the Newtonian potential invariant:

$$\bar{U} = \hat{U}, \tag{8.79}$$

where $\hat{U} = U(\bar{t}, \bar{x}^j + V^j\bar{t})$.

Our results can be used to simplify the general expression for $\bar{U}^j$, as it appears in Eq. (8.54). To keep $\bar{U}^j$ from growing we must set $\dot{R}^j = 0$. The rotation of the coordinate axes described by $R^j(\bar{t})$ must therefore be constant in time, and we choose to eliminate this uninteresting freedom by setting

$$R^j = 0. \tag{8.80}$$

To eliminate the spatially-constant term in $\bar{U}^j$ we set $\partial_{\bar{j}}\beta = \dot{H}^j$, which integrates to $\beta = \beta_0(\bar{t}) + \dot{H}_j\bar{x}^j$, where β_0 and H^j are arbitrary functions of time. We observe that β is a harmonic function, and that its expression is compatible with Eq. (8.63); the transformation is therefore within the class of harmonic transformations. With this result we find that the vector potential transforms as

$$\bar{U}^j = \hat{U}^j - V^j\hat{U} \tag{8.81}$$

under a post-Galilean transformation.

Moving on to $\bar{\Psi}$, as it appears in Eq. (8.58), we find that the removal of the growing term requires $\ddot{H}^j = 0$, so that $\dot{H}^j$ must be a constant vector. This vector is in principle arbitrary, but we choose to restrict the coordinate freedom by making it proportional to V^j. We write it as $\dot{H}^j = \frac{1}{2}V^2V^j$, inserting an arbitrary numerical coefficient of $\frac{1}{2}$ for reasons that will be made clear below, and the factor of V^2 for proper dimensionality. Our choice for $H^j(\bar{t})$ is therefore

$$H^j = \frac{1}{2}V^2V^j\bar{t}. \tag{8.82}$$

To eliminate the spatially-constant term in $\bar{\Psi}$ we must set $\dot{\beta}_0 = -\frac{1}{8}V^4 + \dot{H}_j V^j$. With our previous choice for H^j this is $\dot{\beta}_0 = \frac{3}{8}V^4$, and our final expression for β is

$$\beta = \frac{3}{8}V^4\bar{t} + \frac{1}{2}V^2 V_j \bar{x}^j. \tag{8.83}$$

With all this we find that the post-Newtonian potential transforms as

$$\bar{\Psi} = \hat{\Psi} - 4V_j\hat{U}^j + 2V^2\hat{U} + \biggl(\frac{1}{2}V^2\bar{t} + V_j\bar{x}^j\biggr)\partial_{\bar{t}}\hat{U} - \biggl(\frac{1}{2}V^j V_k\bar{x}^k\biggr)\partial_{\bar{j}}\hat{U} \tag{8.84}$$

under a post-Galilean transformation. We note that the terms involving $\partial_{\bar{t}}\hat{U}$ and $\partial_{\bar{j}}\hat{U}$ are multiplied by quantities that grow linearly with $\bar{r}$. Because $\hat{U}$ decays as $\bar{r}^{-1}$, and its derivatives as $\bar{r}^{-2}$, we see that $\bar{\Psi}$ properly vanishes in the formal limit $\bar{r} \to \infty$.

Collecting results, we find that the post-Galilean transformation is a three-parameter family described by

$$t = \biggl(1 + \frac{1}{2}\frac{V^2}{c^2} + \frac{3}{8}\frac{V^4}{c^4}\biggr)\bar{t} + \frac{1}{c^2}\biggl(1 + \frac{1}{2}\frac{V^2}{c^2}\biggr)V_j\bar{x}^j + O(c^{-6}), \tag{8.85a}$$

$$x^j = \biggl(\delta^j{}_k + \frac{1}{2}\frac{V^j V_k}{c^2}\biggr)\bar{x}^k + \biggl(1 + \frac{1}{2}\frac{V^2}{c^2}\biggr)V^j\bar{t} + O(c^{-4}); \tag{8.85b}$$

the parameters are the three components of the vector V^j. This is nothing but a Lorentz transformation expanded in powers of $(V/c)^2$. The coordinates $(\bar{t}, \bar{x}^j)$ define a frame $\bar{S}$ that is boosted with respect to the original frame S; the boost takes place in the direction of the velocity vector V^j.

Boosted potentials

In the foregoing discussion the boosted potentials $\bar{U}$, $\bar{U}^j$, and $\bar{\Psi}$ were expressed in terms of the “hatted potentials” $\hat{U}$, $\hat{U}^j$, and $\hat{\Psi}$; these, we recall, are the original potentials evaluated at time $t = \bar{t}$ and position $x^j = \bar{x}^j + V^j\bar{t}$. This representation of the transformed potentials was optimal in the context of the general theory of post-Newtonian transformations, as developed in the previous sections. It is not optimal in the restricted context of post-Galilean transformations, because of the schizophrenic nature of the hatted potentials, which live partly in the frame S and partly in the frame $\bar{S}$. An indication that the representation is indeed not optimal comes from our previous expression for $\bar{\Psi}$, which displays a curious and unwanted explicit dependence upon $\bar{t}$ and $\bar{x}^j$.

We therefore proceed differently. We shall (i) postulate plausible expressions for the transformed potentials $\bar{U}$, $\bar{U}^j$, and $\bar{\Psi}$; (ii) relate these to the original potentials U, U^j, and Ψ; and (iii) show that under the transformation of Eqs. (8.85), the transformed metric $\bar{g}_{\alpha\beta}$ keeps the standard post-Newtonian form of Eq. (8.20), with the understanding that the new metric is expressed in terms of the new potentials.

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Our proposed expressions for the transformed potentials are

$$\bar{U}(\bar{t}, \bar{\boldsymbol{x}}) = G \int \frac{\bar{\rho}^*(\bar{t}, \bar{\boldsymbol{x}}')}{|\bar{\boldsymbol{x}} - \bar{\boldsymbol{x}}'|}\, d^3\bar{x}', \tag{8.86a}$$

$$\bar{U}^j(\bar{t}, \bar{\boldsymbol{x}}) = G \int \frac{\bar{\rho}^* \bar{v}^j(\bar{t}, \bar{\boldsymbol{x}}')}{|\bar{\boldsymbol{x}} - \bar{\boldsymbol{x}}'|}\, d^3\bar{x}', \tag{8.86b}$$

$$\bar{\psi}(\bar{t}, \bar{\boldsymbol{x}}) = G \int \frac{\bar{\rho}^*(\frac{3}{2}\bar{v}^2 - \bar{U} + \bar{\Pi} + 3\bar{p}/\bar{\rho}^*)(\bar{t}, \bar{\boldsymbol{x}}')}{|\bar{\boldsymbol{x}} - \bar{\boldsymbol{x}}'|}\, d^3\bar{x}', \tag{8.86c}$$

$$\bar{X}(\bar{t}, \bar{\boldsymbol{x}}) = G \int \rho^*(\bar{t}, \bar{\boldsymbol{x}}')|\bar{\boldsymbol{x}} - \bar{\boldsymbol{x}}'|\, d^3\bar{x}', \tag{8.86d}$$

and the transformed post-Newtonian potential is $\bar{\Psi} = \bar{\psi} + \frac{1}{2}\partial_{\bar{t}\bar{t}}\bar{X}$. These expressions are natural: the new potentials are defined just as the old potentials in terms of the boosted coordinates and the fluid variables $\bar{\rho}^*$, $\bar{p}$, $\bar{\Pi}$, and $\bar{\boldsymbol{v}}$ that would be measured in the frame $\bar{S}$ instead of the original frame S. It is useful to introduce transformed versions of the auxiliary potentials listed in Eqs. (8.8); in terms of these we have $\bar{\Psi} = 2\bar{\Phi}_1 - \bar{\Phi}_2 + \bar{\Phi}_3 + 4\bar{\Phi}_4 - \frac{1}{2}\bar{\Phi}_5 - \frac{1}{2}\bar{\Phi}_6$.

Our first task is to express the old Newtonian potential $U(t, \boldsymbol{x})$ in terms of the new potentials. This is not entirely straightforward. A major source of subtlety is the important fact that in Eqs. (8.86), the integration variables $\bar{\boldsymbol{x}}'$ describe the position of a fluid element at time $\bar{t}$, the same time at which the potentials are being evaluated. The spacetime events P and P', respectively labeled by the coordinates $(\bar{t}, \bar{\boldsymbol{x}})$ and $(\bar{t}, \bar{\boldsymbol{x}}')$, are simultaneous in the frame $\bar{S}$. But they are not simultaneous in the original frame S, and we must take this property carefully into account.

We examine the situation in the original frame S (see Fig. 8.1). The figure shows a spacetime diagram in which we display the field point P as well as the world line of a selected fluid element. Two events are shown on this world line: the source point Q' is simultaneous with P in the frame S, while P' is simultaneous with P in the frame $\bar{S}$. In the frame S the coordinates of P are $(t, \boldsymbol{x})$, the coordinates of Q' are $(t, \boldsymbol{x}')$, and the coordinates of P' are $(\tau, \boldsymbol{\xi})$. In the frame $\bar{S}$ the coordinates of P are $(\bar{t}, \bar{\boldsymbol{x}})$, the coordinates of Q' are $(\bar{\tau}, \bar{\boldsymbol{\xi}})$, and the coordinates of P' are $(\bar{t}, \bar{\boldsymbol{x}}')$. Note that the coordinates $(\tau, \boldsymbol{\xi})$ and $(\bar{\tau}, \bar{\boldsymbol{\xi}})$ refer to different events in spacetime. In the frame S the world line is described by the time-dependent position vector $\boldsymbol{r}$; we have that $\boldsymbol{x}' := \boldsymbol{r}(t)$ and $\boldsymbol{\xi} := \boldsymbol{r}(\tau)$. In the frame $\bar{S}$ the world line is described by $\bar{\boldsymbol{r}}$, and we have that $\bar{\boldsymbol{x}}' := \bar{\boldsymbol{r}}(\bar{t})$ and $\bar{\boldsymbol{\xi}} := \bar{\boldsymbol{r}}(\bar{\tau})$. In the frame S the velocity of the fluid element at Q' is $\boldsymbol{v}' := \dot{\boldsymbol{r}}(t)$, while in $\bar{S}$ the velocity of the fluid element at P' is $\bar{\boldsymbol{v}}' := \dot{\bar{\boldsymbol{r}}}(\bar{t})$; the overdots indicate differentiation with respect to the relevant time variable.

The coordinates of the field point P transform as in Eq. (8.85). The coordinates of the source point Q' transform as

$$t = \left(1 + \frac{1}{2}\frac{V^2}{c^2} + \frac{3}{8}\frac{V^4}{c^4}\right)\bar{\tau} + \frac{1}{c^2}\left(1 + \frac{1}{2}\frac{V^2}{c^2}\right)V_j\bar{\xi}^j + O(c^{-6}), \tag{8.87a}$$

$$x'^j = \left(\delta^j{}_k + \frac{1}{2}\frac{V^jV_k}{c^2}\right)\bar{\xi}^k + \left(1 + \frac{1}{2}\frac{V^2}{c^2}\right)V^j\bar{\tau} + O(c^{-4}). \tag{8.87b}$$

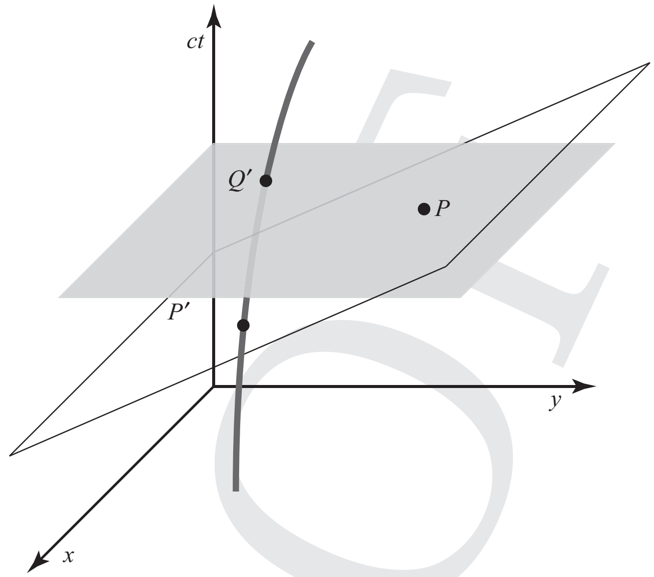

Fig. 8.1 World line of a selected fluid element viewed in the frame S. The grey plane is a hypersurface $t =$ constant, and the white plane is a hypersurface $\bar{t} =$ constant. The field point P is simultaneous with Q' in S, but it is simultaneous with P' in the frame $\bar{S}$.

We want to express $\boldsymbol{x}'$ in terms of $\bar{t}$, $\bar{\boldsymbol{x}}$, and $\bar{\boldsymbol{x}}'$, and this requires elimination of $\bar{\tau}$ and $\bar{\boldsymbol{\xi}}$ from Eqs. (8.87b). To achieve this we equate the t of Eqs. (8.85) with the t of Eqs. (8.87) and deduce that $\bar{\tau} = \bar{t} + c^{-2} V_j (\bar{x} - \bar{\xi})^j + O(c^{-4})$. We substitute this into the world-line equation $\bar{\boldsymbol{\xi}} = \bar{\boldsymbol{r}}(\bar{\tau})$ and get $\bar{\xi}^j = \bar{x}'^j + c^{-2} \bar{v}'^j V_k (\bar{x} - \bar{\xi})^k + O(c^{-4})$. Collecting results, we have obtained

$$\bar{\tau} = \bar{t} + \frac{V_k}{c^2} (\bar{x} - \bar{x}')^k + O(c^{-4}), \tag{8.88a}$$

$$\bar{\xi}^j = \bar{x}'^j + \frac{v'^j V_k}{c^2} (\bar{x} - \bar{x}')^k + O(c^{-4}). \tag{8.88b}$$

These are the coordinates of Q' in the frame $\bar{S}$, expressed in terms of the coordinates of both P and P'. Making the substitution in Eq. (8.87b), we arrive at

$$x'^j = \bar{x}'^j + V^j \bar{t} + \frac{1}{c^2} \big(\bar{v}'^j + V^j \big) V_k (\bar{x} - \bar{x}')^k + \frac{V^j}{2c^2} \big(V^2 \bar{t} + V_k \bar{x}'^k \big) + O(c^{-4}), \tag{8.89}$$

the desired relation between $\boldsymbol{x}'$ and the coordinates of P and P' in the frame $\bar{S}$. To this equation we can adjoin

$$v'^j = \bar{v}'^j + V^j + O(c^{-2}), \tag{8.90}$$

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the law of composition of velocities (truncated to the leading, Newtonian order). It follows from Eqs. (8.85) and (8.89) that

$$\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} = \frac{1}{|\bar{\boldsymbol{x}}-\bar{\boldsymbol{x}}'|} + \frac{1}{c^2}\Big(\bar{v}'_j V_k + \frac{1}{2}V_j V_k\Big)\frac{(\bar{x}-\bar{x}')^j(\bar{x}-\bar{x}')^k}{|\bar{\boldsymbol{x}}-\bar{\boldsymbol{x}}'|^3} + O(c^{-4}), \tag{8.91}$$

and this is an important ingredient that enters the transformation of the Newtonian potential.

Another important ingredient is the statement that $\rho^* d^3x$ is invariant under the post-Galilean transformation. We express this as

$$\rho^*(t,\boldsymbol{x}')\,d^3x' = \bar{\rho}^*(\bar{t},\bar{\boldsymbol{x}}')\,d^3\bar{x}'. \tag{8.92}$$

The invariance of $dm := \rho^* d^3x$ reflects the simple fact that dm is the conserved rest-mass of a fluid element. Because this cannot be altered by a coordinate transformation, we have that $dm(Q') = d\bar{m}(Q')$, or $dm(t,\boldsymbol{x}') = d\bar{m}(\bar{\tau},\bar{\boldsymbol{\xi}})$. And because $d\bar{m}$ does not change as we follow the motion of the fluid element, we also have that $d\bar{m}(Q') = d\bar{m}(P')$, or $d\bar{m}(\bar{\tau},\bar{\boldsymbol{\xi}}) = d\bar{m}(\bar{t},\bar{\boldsymbol{x}}')$. We therefore arrive at Eq. (8.92), which is just the combined statement that $dm(Q') = d\bar{m}(P')$.

More formally, $dm(Q') = d\bar{m}(Q')$ is a consequence of the facts that (i) the proper mass density ρ is a scalar quantity; (ii) the spacetime volume element $\sqrt{-g}dtd^3x$ is invariant under a coordinate transformation; and (iii) the element of proper time $d\lambda$ along the world line also is an invariant. From all this it follows that $\rho\sqrt{-g}(dt/d\lambda)d^3x =: \rho^*\, d^3x$ is invariant under the transformation. On the other hand, that $d\bar{m}(Q') = \mathrm{d}\bar{m}(P')$ follows formally from an application of the continuity equation, Eq. (7.3), to a single fluid element. The formal route also gives rise to the statement of Eq. (8.92).

When we substitute Eqs. (8.91) and (8.92) into the integral definition of the old Newtonian potential $U(t,\boldsymbol{x})$, we find that it can be expressed in terms of the new potentials as

$$U = \Big(1 + \frac{V^2}{2c^2}\Big)\bar{U} + \frac{V^j}{c^2}\bar{\Phi}_j - \frac{V^jV^k}{2c^2}\partial_{\bar{j}\bar{k}}\bar{X} + O(c^{-4}). \tag{8.93}$$

Here $\bar{\Phi}_j$ is the transformed version of the auxiliary potential defined by Eq. (8.8), and we make use of the identity of Eq. (8.11).

The remaining potentials transform in an analogous way. In fact, their transformation properties are much easier to identify, because here we do not need to calculate the correction terms of order c^{-2}; these impact the metric beyond the first post-Newtonian order. Taking into account Eq. (8.90) and the fact that p and Π transform as scalar quantities, we quickly obtain

$$U^j = \bar{U}^j + V^j\bar{U} + O(c^{-2}), \tag{8.94}$$

as well as

$$\Phi_1 = \bar{\Phi}_1 + 2V^j\bar{U}_j + V^2\bar{U} + O(c^{-2}), \tag{8.95a}$$
$$\Phi_2 = \bar{\Phi}_2 + O(c^{-2}), \tag{8.95b}$$
$$\Phi_3 = \bar{\Phi}_3 + O(c^{-2}), \tag{8.95c}$$
$$\Phi_4 = \bar{\Phi}_4 + O(c^{-2}), \tag{8.95d}$$
$$\Phi_5 = \bar{\Phi}_5 + O(c^{-2}), \tag{8.95e}$$
$$\Phi_6 = \bar{\Phi}_6 + 2V^j\bar{\Phi}_j + V^2\bar{U} - V^jV^k\partial_{\bar{j}\bar{k}}\bar{X} + O(c^{-2}). \tag{8.95f}$$

These equations imply that the post-Newtonian potential transforms as

$$\Psi = \bar{\Psi} + V^j\big(4\bar{U}_j - \bar{\Phi}_j\big) + \frac{3}{2}V^2\bar{U} + \frac{1}{2}V^jV^k\partial_{\bar{j}\bar{k}}\bar{X} + O(c^{-2}) \tag{8.96}$$

under a post-Galilean transformation.

Our final task is to verify that the transformed metric $\bar{g}_{\alpha\beta}$ takes the standard post-Newtonian form of Eq. (8.20) when it is expressed in terms of the transformed potentials $\bar{U}$, $\bar{U}^j$, and $\bar{\Psi}$. This is straightforward. We first specialize the general transformation equations (8.34) to the post-Galilean case of Eqs. (8.85) and get the components $\bar{g}_{00} = -1 + 2c^{-2}U + 2c^{-4}(\Psi - U^2 - 4V^jU_j + 2V^2U) + O(c^{-6})$, $\bar{g}_{0j} = -4c^{-3}(U_j - V_jU) + O(c^{-5})$, and $\bar{g}_{jk} = \delta_{jk}(1 + 2c^{-2}U) + O(c^{-4})$ for the new metric tensor. This is still expressed in terms of the old potentials, and we complete the calculation by involving Eqs. (8.93), (8.94), and (8.96). Our final result is

$$\bar{g}_{00} = -1 + \frac{2}{c^2}\bar{U} + \frac{2}{c^4}\big(\bar{\Psi} - \bar{U}^2\big) + O(c^{-6}), \tag{8.97a}$$

$$\bar{g}_{0j} = -\frac{4}{c^3}\bar{U}_j + O(c^{-5}), \tag{8.97b}$$

$$\bar{g}_{jk} = \left(1 + \frac{2}{c^2}\bar{U}\right)\delta_{jk} + O(c^{-4}), \tag{8.97c}$$

the statement that the transformed metric does indeed take the standard post-Newtonian form in terms of the proposed new potentials of Eqs. (8.86).

This completes our discussion of post-Galilean transformations. To sum up, we have established that a post-Galilean transformation describes a boost from a frame S to a new frame $\bar{S}$ that moves relative to S with a constant velocity $\boldsymbol{V}$. In this new frame the metric keeps its standard post-Newtonian form, but the potentials are now defined by Eqs. (8.86); they refer to the fluid variables $\bar{\rho}^*$, $\bar{p}$, $\bar{\Pi}$, and $\bar{\boldsymbol{v}}$ that are measured in the new frame.

8.3.7 Pure-gauge transformations

Another interesting subclass of post-Newtonian transformations is obtained by setting

$$A = r^j = H^j = R^j = 0 \tag{8.98}$$

and retaining only the freedom contained in β. This class of transformations is described by

$$t = \bar{t} + \frac{1}{c^4}\beta(\bar{t}, \bar{x}^j) + O(c^{-6}), \tag{8.99a}$$

$$x^j = \bar{x}^j + O(c^{-4}), \tag{8.99b}$$

and the potentials change according to

$$\bar{U} = U, \tag{8.100a}$$

$$\bar{U}^j = U^j + \frac{1}{4}\partial_{\bar{j}}\beta, \tag{8.100b}$$

$$\bar{\Psi} = \Psi - \partial_{\bar{t}}\beta. \tag{8.100c}$$

In this case we no longer need to distinguish between the hatted potentials and their original expressions. Equations (8.100) take the appearance of an electromagnetic-type gauge transformation that links the potentials U^j and Ψ. We refer to this subclass of transformations as the *pure-gauge transformations* of post-Newtonian theory. When β is a harmonic function, the gauge transformation converts a set of harmonic coordinates to another set of harmonic coordinates.

The transformation may be exploited to remove the superpotential term from g_{00} and put it instead in g_{0j}. We refer to the decomposition of Eq. (8.2), and to eliminate the term $\frac{1}{2}\partial_{tt}X$ from g_{00} we choose

$$\beta = \frac{1}{2}\partial_{\bar{t}}X. \tag{8.101}$$

Note that this is not a harmonic function (because $\nabla^2 X = 2U$), so that the transformation does not preserve the harmonic coordinate condition. With this expression for β we find that the new metric is given by

$$\bar{g}_{00} = -1 + \frac{2}{c^2}U + \frac{2}{c^4}\left(\psi - U^2\right) + O(c^{-6}), \tag{8.102a}$$

$$\bar{g}_{0j} = -\frac{4}{c^3}U_j - \frac{1}{2c^3}\partial_{\bar{t}\bar{j}}X + O(c^{-5}), \tag{8.102b}$$

$$\bar{g}_{jk} = \left(1 + \frac{2}{c^2}U\right)\delta_{jk} + O(c^{-4}). \tag{8.102c}$$

This choice of coordinate system defines the so-called *standard gauge* of post-Newtonian theory. As we have pointed out in the introductory section, this choice of gauge was popularized by Chandrasekhar, and it was once widely utilized by researchers in the post-Newtonian community. Like most current workers in the field, however, we prefer to use the harmonic gauge, and we have made this choice consistently throughout the book, except in Chapter 13 where we examine alternative theories of gravity. To be sure, the choice of gauge is mostly a matter of taste and convenience. But there are, nevertheless, important advantages in using the harmonic coordinates: it is in this gauge that post-Newtonian theory can naturally be embedded within the wider context of post-Minkowskian theory. And as we explained back in Sec. 8.2, it is by doing this that the foundations of post-Newtonian theory can be made secure.

8.4 Post-Newtonian hydrodynamics

8.4.1 Introduction

The dynamics of perfect fluids has been a recurring topic in this book. We examined this first in the context of Newtonian physics in Sec. 1.4, we gave the theory a special-relativistic formulation in Sec. 4.2, and we promoted this to curved spacetime in Sec. 5.3. In Sec. 7.1.1 we introduced the variables $\{\rho^*, p, \Pi, \boldsymbol{v}\}$ and incorporated slowly-moving fluids within the post-Minkowskian approximation.

In Sec. 7.1.1 we saw that the behavior of a perfect fluid is governed by the continuity equation

$$\partial_t \rho^* + \partial_j(\rho^* v^j) = 0, \tag{8.103}$$

and in Sec. 7.3.2 we got re-acquainted with the Euler equation of Chapter 1,

$$\rho^* \frac{dv^j}{dt} = \rho^* \partial_j U - \partial_j p + O(c^{-2}). \tag{8.104}$$

We recall that $\rho^* := \sqrt{-g}\gamma\rho$, with $\gamma := u^0/c$, is the conserved mass density, and at Newtonian order there is no distinction between this and the proper density ρ; v^j is the fluid's velocity field, defined with respect to the time coordinate t, p is the pressure, and $d/dt = \partial_t + v^k\partial_k$ is the Lagrangian time derivative. We recall also that the exact statement of the first law of thermodynamics for perfect fluids is $d\Pi = (p/\rho^2)\,d\rho$, which we write as

$$\frac{d\Pi}{dt} = \frac{p}{\rho^{*2}}\frac{d\rho^*}{dt} + O(c^{-2}). \tag{8.105}$$

Here Π is the internal energy of a fluid element divided by its mass.

In this section we calculate the post-Newtonian corrections to Euler's equation (8.104). In addition, we derive expressions for the fluid's conserved mass-energy M, its total momentum P^j, and its center-of-mass position R^j. We shall not alter Eq. (8.103), which is exact, and we shall not need the $O(c^{-2})$ corrections to Eq. (8.105). In Chapter 9 we apply these results to situations in which the fluid breaks up into a number of separated components; this defines the post-Newtonian N-body problem.

8.4.2 Energy-momentum conservation

The components of the energy-momentum tensor of a perfect fluid were listed back in Eq. (8.7). The equation of energy-momentum conservation is

$$0 = \nabla_\beta T^{\alpha\beta} = \partial_\beta T^{\alpha\beta} + \Gamma^\alpha_{\mu\beta}T^{\mu\beta} + \Gamma^\beta_{\mu\beta}T^{\alpha\mu}\,, \tag{8.106}$$

and this can be simplified if we recall from Sec. 5.2 that $\Gamma^\beta_{\mu\beta} = (-g)^{-1/2}\partial_\beta(-g)^{1/2}$. We therefore have

$$0 = \partial_\beta\big(\sqrt{-g}T^{\alpha\beta}\big) + \Gamma^\alpha_{\beta\gamma}\big(\sqrt{-g}T^{\beta\gamma}\big)\,, \tag{8.107}$$

and this form of the conservation equation is optimal for the following computations. We recall that the square root of the metric determinant is given by $\sqrt{-g} = 1 + 2c^{-2}U + O(c^{-4})$.

The zeroth component of Eq. (8.107) gives rise to a statement of energy conservation. When fully expanded the equation is

$$\begin{aligned} 0 &= \frac{1}{c}\partial_t\big(\sqrt{-g}T^{00}\big) + \partial_j\big(\sqrt{-g}T^{0j}\big) \\ &\quad + \Gamma^0_{00}\big(\sqrt{-g}T^{00}\big) + 2\Gamma^0_{0j}\big(\sqrt{-g}T^{0j}\big) + \Gamma^0_{jk}\big(\sqrt{-g}T^{jk}\big)\,, \end{aligned} \tag{8.108}$$

and this becomes

$$\begin{aligned} 0 &= c\Big[\partial_t\rho^* + \partial_j(\rho^* v^j)\Big] \\ &\quad + \frac{1}{c}\bigg\{\partial_t\bigg[\rho^*\Big(\frac{1}{2}v^2 + U + \Pi\Big)\bigg] + \partial_j\bigg[\rho^* v^j\Big(\frac{1}{2}v^2 + U + \Pi\Big)\bigg] \\ &\qquad + \partial_j(pv^j) - \rho^*\partial_t U - 2\rho^* v^j\partial_j U\bigg\} + O(c^{-3}), \end{aligned} \tag{8.109}$$

after inserting the components of $T^{\alpha\beta}$ and the Christoffel symbols of Eq. (8.15). At order c we recover the continuity equation (8.103), and at order c^{-1} we get

$$0 = \rho^*\partial_t\Big(\frac{1}{2}v^2 + \Pi\Big) + \rho^* v^j\partial_j\Big(\frac{1}{2}v^2 + \Pi\Big) + \partial_j(pv^j) - \rho^* v^j\partial_j U \tag{8.110}$$

after simplification. This is the equation that expresses the local conservation of energy within the fluid.

The spatial components of Eq. (8.107) provide a statement of momentum conservation. We write the equation in fully expanded form as

$$\begin{aligned} 0 &= \frac{1}{c}\partial_t\big(\sqrt{-g}T^{0j}\big) + \partial_k\big(\sqrt{-g}T^{jk}\big) \\ &\quad + \Gamma^j_{00}\big(\sqrt{-g}T^{00}\big) + 2\Gamma^j_{0k}\big(\sqrt{-g}T^{0k}\big) + \Gamma^j_{kn}\big(\sqrt{-g}T^{kn}\big)\,, \end{aligned} \tag{8.111}$$

and we eventually arrive at

$$\begin{aligned} 0 &= \partial_t(\mu\rho^* v^j) + \partial_k(\mu\rho^* v^j v^k) + \partial_j p - \rho^*\partial_j U - \frac{\rho^*}{c^2}\Big(\frac{3}{2}v^2 - 3U + \Pi + p/\rho^*\Big)\partial_j U \\ &\quad + \frac{\rho^*}{c^2}\Big[2v_j(\partial_t U + v^k\partial_k U) - 4\partial_t U_j - 4v^k(\partial_k U_j - \partial_j U_k) - \partial_j\Psi\Big] + O(c^{-4}) \end{aligned} \tag{8.112}$$

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after some algebra and simplification. We have introduced

$$\mu := 1 + \frac{1}{c^2}\left(\frac{1}{2}v^2 + U + \Pi + p/\rho^*\right) + O(c^{-4})\,, \tag{8.113}$$

and Eq. (8.112) expresses the local conservation of momentum within the fluid.

8.4.3 Post-Newtonian Euler equation

We next work on Eq. (8.112) and bring it to the form of a relativistic generalization of Eq. (8.104). We begin with the observation that the first two terms on the right-hand side of Eq. (8.112) can be expressed as

$$\partial_t(\mu\rho^* v^j) + \partial_k(\mu\rho^* v^j v^k) = \mu\rho^*\frac{dv^j}{dt} + \rho^* v^j\frac{d\mu}{dt} \tag{8.114}$$

after making use of the continuity equation (8.103). If we make the substitution in Eq. (8.112) and truncate the result at Newtonian order, we recover

$$\rho^*\frac{dv^j}{dt} = \rho^*\partial_j U - \partial_j p + O(c^{-2})\,, \tag{8.115}$$

the correct expression of Euler's equation. We are therefore on the right track, and we may now retrieve the neglected terms of order c^{-2}.

Differentiation of Eq. (8.113) yields

$$\frac{d\mu}{dt} = \frac{1}{c^2}\left(v_j\frac{dv^j}{dt} + \frac{dU}{dt} + \frac{d\Pi}{dt} + \frac{1}{\rho^*}\frac{dp}{dt} - \frac{p}{\rho^{*2}}\frac{d\rho^*}{dt}\right) + O(c^{-4}), \tag{8.116}$$

and this becomes

$$\frac{d\mu}{dt} = \frac{1}{c^2}\left(\partial_t U + 2v^k\partial_k U + \frac{1}{\rho^*}\partial_t p\right) + O(c^{-4}) \tag{8.117}$$

after insertion of Euler's equation and Eq. (8.105).

Substitution of Eqs. (8.114) and (8.117) into Eq. (8.112) produces

$$\begin{aligned}\mu\rho^*\frac{dv^j}{dt} &= -\partial_j p + \rho^*\partial_j U - \frac{1}{c^2}v^j\partial_t p + \frac{1}{c^2}\rho^*\left(\frac{3}{2}v^2 - 3U + \Pi + \frac{p}{\rho^*}\right)\partial_j U \\ &\quad - \frac{1}{c^2}\rho^*\Big[v^j\big(3\partial_t U + 4v^k\partial_k U\big) - 4\partial_t U_j - 4v^k\big(\partial_k U_j - \partial_j U_k\big) - \partial_j\Psi\Big] + O(c^{-4}),\end{aligned} \tag{8.118}$$

and this becomes

$$\begin{aligned}\rho^* \frac{dv^j}{dt} &= -\partial_j p + \rho^* \partial_j U \\ &\quad + \frac{1}{c^2}\biggl[\biggl(\frac{1}{2}v^2 + U + \Pi + \frac{p}{\rho^*}\biggr)\partial_j p - v^j \partial_t p\biggr] \\ &\quad + \frac{1}{c^2}\rho^*\Bigl[(v^2 - 4U)\partial_j U - v^j\bigl(3\partial_t U + 4v^k\partial_k U\bigr) \\ &\qquad + 4\partial_t U_j + 4v^k\bigl(\partial_k U_j - \partial_j U_k\bigr) + \partial_j \Psi\Bigr] \\ &\quad + O(c^{-4})\end{aligned} \tag{8.119}$$

after multiplication of each side by μ^{-1}. Equation (8.119) is the post-Newtonian version of Euler's equation. This equation, together with the continuity equation (8.103) and an equation of state relating the pressure, density, and internal energy, completely determines the behavior of a slowly-moving fluid in a weak gravitational field.

8.4.4 Interlude: Integral identities

We now interrupt the main development and establish a number of identities that will be required in the following discussion. We first derive the results displayed back in Eqs. (8.10), and next we shall prove the integral identities

$$\int \rho^* \partial_j U \, d^3x = 0 \,, \tag{8.120a}$$

$$\int \rho^* U^j \, d^3x = \int \rho^* U v^j \, d^3x \,, \tag{8.120b}$$

$$\int \rho^* \partial_j \psi \, d^3x = -\int \rho^* \bigl(\tfrac{3}{2}v^2 - U + \Pi + 3p/\rho^*\bigr)\partial_j U \, d^3x \,, \tag{8.120c}$$

$$\int \rho^* v^j \partial_j U \, d^3x = \frac{1}{2}\frac{d}{dt}\int \rho^* U \, d^3x \,, \tag{8.120d}$$

$$\int \rho^* v^k \partial_j U_k \, d^3x = 0 \,, \tag{8.120e}$$

$$\int \rho^* v^k \partial_j \Phi_k \, d^3x = 0 \,, \tag{8.120f}$$

$$\int \rho^* x^j \bigl(\partial_t U - v^k \partial_k U\bigr) \, d^3x = \int \rho^* \Phi^j \, d^3x \,. \tag{8.120g}$$

The potentials U, U^j, ψ, and X are expressed in terms of the fluid variables in Eqs. (8.4); the auxiliary potential Φ^j was introduced in Eq. (8.8).

As a consequence of Eqs. (8.10) and (8.120b) we also find that

$$\int \rho^* \bigl(U v_j + \partial_{tj} X\bigr) \, d^3x = \int \rho^* \Phi_j \, d^3x \,. \tag{8.121}$$

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And combining Eqs. (8.10), (8.120e), and (8.120f) yields

$$\int \rho^* v^k \partial_{tjk} X \, d^3x = 0 \,, \tag{8.122}$$

another useful integral identity.

Box 8.4 Integration and time differentiation

The integral tricks reviewed here were first introduced back in Sec. 1.4.3. We recall that

$$\frac{d}{dt} \int \rho^* f(t, \boldsymbol{x}) \, d^3x = \int \rho^* \frac{df}{dt} \, d^3x \tag{1}$$

comes as an immediate consequence of the continuity equation (8.103); here ρ^* is a function of t and $\boldsymbol{x}$, f is an arbitrary function of its arguments, and $df/dt = \partial_t f + v^k \partial_k f$ is its total time derivative. We generalize this identity by allowing f to be a function of $\boldsymbol{x}'$ also, and we define $F(t, \boldsymbol{x}) := \int \rho^{*\prime} f(t, \boldsymbol{x}, \boldsymbol{x}') \, d^3x'$, with $\rho^{*\prime}$ standing for $\rho^*(t, \boldsymbol{x}')$. If we keep $\boldsymbol{x}$ fixed in this equation, Eq. (1) tells us that

$$\partial_t F = \int \rho^{*\prime} \big(\partial_t f + v'^k \partial_{k'} f \big) \, d^3x' \,, \tag{2}$$

where v'^k is the velocity field expressed as a function of t and $\boldsymbol{x}'$, and $\partial_{k'}$ indicates partial differentiation with respect to the primed coordinates. The total time derivative of F is $\partial_t F + v^k \partial_k F$, and this can be expressed as $dF/dt = \int \rho^{*\prime} (\partial_t f + v^k \partial_k f + v'^k \partial_{k'} f) \, d^3x'$. The quantity within brackets is recognized as the total time derivative of the function $f(t, \boldsymbol{x}, \boldsymbol{x}')$, and we write our identity as

$$\frac{dF}{dt} = \int \rho^{*\prime} \frac{df}{dt} \, d^3x' \,, \tag{3}$$

with $df/dt = \partial_t f + v^k \partial_k f + v'^k \partial_{k'} f$. Finally, we define the function $\mathcal{F}(t) := \int \rho^* F(t, \boldsymbol{x}) \, d^3x = \int \rho^* \rho^{*\prime} f(t, \boldsymbol{x}, \boldsymbol{x}') \, d^3x' d^3x$ and insert F in place of f within Eq. (1). After also using Eq. (3) we obtain

$$\frac{d\mathcal{F}}{dt} = \int \rho^* \rho^{*\prime} \frac{df}{dt} \, d^3x' d^3x \,, \tag{4}$$

with df/dt defined as in Eq. (3).

To establish these results we rely on the integral tricks reviewed in Box 8.4. To obtain Eqs. (8.10) we first differentiate $X = G \int \rho^{*\prime} |\boldsymbol{x} - \boldsymbol{x}'| \, d^3x'$ with respect to time. Using Eq. (2) of Box 8.4, we find $\partial_t X = G \int \rho^{*\prime} v'^k \partial_{k'} |\boldsymbol{x} - \boldsymbol{x}'| \, d^3x'$, or

$$\partial_t X = -G \int \rho^{*\prime} v'_k \frac{(x - x')^k}{|\boldsymbol{x} - \boldsymbol{x}'|} \, d^3x' \,. \tag{8.123}$$

We next differentiate this with respect to x^j and get

$$\partial_{tj} X = -G \int \rho^{*\prime} v'_k \partial_j \frac{(x - x')^k}{|\boldsymbol{x} - \boldsymbol{x}'|} \, d^3x' \,, \tag{8.124}$$

which becomes

$$\partial_{tj} X = -G \int \frac{\rho^{*\prime} v'_j}{|\boldsymbol{x}-\boldsymbol{x}'|} d^3x' + G \int \rho^{*\prime} v'_k \frac{(x-x')^j (x-x')^k}{|\boldsymbol{x}-\boldsymbol{x}'|^3} \tag{8.125}$$

after evaluation of the partial derivative. In view of the definitions for U_j and Φ_j, this is just the first of Eqs. (8.10). If we differentiate instead with respect to time, we get

$$\partial_{tt} X = -G \int \rho^{*\prime} \left[\frac{dv'_k}{dt} \frac{(x-x')^k}{|\boldsymbol{x}-\boldsymbol{x}'|} + v'_k v'^j \partial_{j'} \frac{(x-x')^k}{|\boldsymbol{x}-\boldsymbol{x}'|} \right] d^3x' .$$

The second term within the integral is handled as before, but to evaluate the first we need an expression for $\rho^{*\prime} dv'_k/dt$. This was obtained back in Eq. (8.119), in the form of the post-Newtonian Euler equation, and we may substitute this here. But since X is a post-Newtonian potential that always appears with an accompanying factor of c^{-2}, it is appropriate to truncate the Euler equation to its Newtonian form, and we therefore use $\rho^{*\prime} dv'_k/dt = -\partial_{k'} p' + \rho^{*\prime} \partial_{k'} U' + O(c^{-2})$. Our expression for $\partial_{tt} X$ becomes

$$\begin{aligned} \partial_{tt} X = {} & G \int \partial_{k'} p' \frac{(x-x')^k}{|\boldsymbol{x}-\boldsymbol{x}'|} d^3x' - G \int \rho^{*\prime} \partial_{k'} U' \frac{(x-x')^k}{|\boldsymbol{x}-\boldsymbol{x}'|} d^3x' \\ & + G \int \frac{\rho^{*\prime} v'^2}{|\boldsymbol{x}-\boldsymbol{x}'|} d^3x' - G \int \rho^{*\prime} v'_j v'_k \frac{(x-x')^j (x-x')^k}{|\boldsymbol{x}-\boldsymbol{x}'|^3} d^3x' , \end{aligned} \tag{8.126}$$

and the first term can be changed to $2G \int p' |\boldsymbol{x}-\boldsymbol{x}'|^{-1} d^3x'$ by integration by parts. Taking into account the definitions of Eqs. (8.8), we see that this is just the second of Eqs. (8.10).

Moving on to the integral identities of Eqs. (8.120), we first differentiate $U = \int \rho^{*\prime} |\boldsymbol{x} - \boldsymbol{x}'|^{-1} d^3x'$ with respect to x^j, multiply by ρ^*, and integrate. This gives

$$\int \rho^* \partial_j U \, d^3x = G \int \rho^* \rho^{*\prime} \frac{(x-x')^j}{|\boldsymbol{x}-\boldsymbol{x}'|^3} d^3x' d^3x,$$

and by switching the identities of the integration variables ($\boldsymbol{x} \leftrightarrow \boldsymbol{x}'$), we may also express the right-hand side as $G \int \rho^{*\prime} \rho^* (x'-x)^j |\boldsymbol{x}' - \boldsymbol{x}|^{-3} d^3x d^3x'$. This is equal and opposite to the original integral, and we conclude that the integral vanishes; Eq. (8.120a) is thus established. (We frequently exploit the "switch trick" in the following manipulations.)

With $U^j = G \int \rho^{*\prime} v'^j |\boldsymbol{x}-\boldsymbol{x}'|^{-1} d^3x'$ we find that the integral of $\rho^* U^j$ is given by

$$\int \rho^* U^j \, d^3x = G \int \frac{\rho^* \rho^{*\prime} v'^j}{|\boldsymbol{x}-\boldsymbol{x}'|} d^3x' d^3x. \tag{8.127}$$

Applying the switch trick, we write the right-hand side as $G \int \rho^{*\prime} \rho^* v^j |\boldsymbol{x}-\boldsymbol{x}'|^{-1} d^3x d^3x'$, which we recognize as the integral of $\rho^* U v^j$ with respect to d^3x. Equation (8.120b) is thus established, and Eq. (8.120c) is obtained with very similar manipulations.

Moving on to Eq. (8.120d), we write

$$\int \rho^* v^j \partial_j U \, d^3x = G \int \rho^* \rho^{*\prime} v^j \partial_j |\boldsymbol{x}-\boldsymbol{x}'|^{-1} d^3x' d^3x, \tag{8.128}$$

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and note that we can re-express the right-hand side as $G\int \rho^{*\prime}\rho^* v'^j \partial_{j'}|\boldsymbol{x}-\boldsymbol{x}'|^{-1}\, d^3x d^3x'$. Adding the expressions and multiplying by $\frac{1}{2}$, we have

$$\frac{1}{2}G\int \rho^*\rho^{*\prime}\big(v^j\partial_j + v'^j\partial_{j'}\big)|\boldsymbol{x}-\boldsymbol{x}'|^{-1}\, d^3x'd^3x,$$

and according to Eq. (4) of Box 8.4, this is

$$\frac{1}{2}G\frac{d}{dt}\int \rho^*\rho^{*\prime}|\boldsymbol{x}-\boldsymbol{x}'|^{-1}\, d^3x'd^3x.$$

We recognize this as the time derivative of $\frac{1}{2}\int \rho^* U\, d^3x$, and we have established Eq. (8.120d).

The identities of Eqs. (8.120e) and (8.120f) are obtained almost immediately by writing out the integrals and exploiting the switch trick. For Eq. (8.120g) we differentiate the Newtonian potential and construct the integrals that appear on the left-hand side. We have

$$\int \rho^* x^j \partial_t U\, d^3x = G\int \rho^*\rho^{*\prime} v'_k x^j \frac{(x-x')^k}{|\boldsymbol{x}-\boldsymbol{x}'|^3}\, d^3x'd^3x \tag{8.129}$$

and

$$\begin{aligned}\int \rho^* x^j v^k \partial_k U\, d^3x &= -G\int \rho^*\rho^{*\prime} v_k x^j \frac{(x-x')^k}{|\boldsymbol{x}-\boldsymbol{x}'|^3}\, d^3x'd^3x \\ &= G\int \rho^{*\prime}\rho^* v'_k x'^j \frac{(x-x')^k}{|\boldsymbol{x}-\boldsymbol{x}'|^3}\, d^3xd^3x',\end{aligned} \tag{8.130}$$

and combining the results produces

$$\int \rho^* x^j(\partial_t U - v^k\partial_k U)\, d^3x = G\int \rho^*\rho^{*\prime} v'_k \frac{(x-x')^j(x-x')^k}{|\boldsymbol{x}-\boldsymbol{x}'|^3}\, d^3x'd^3x. \tag{8.131}$$

In view of the definition of Φ^j provided by Eq. (8.8), this is just Eq. (8.120g).

8.4.5 Conservation of mass-energy

We now resume our development of post-Newtonian hydrodynamics. In this and the following sections we obtain expressions for the total mass-energy M and momentum P^j of a fluid system, as well as an expression for the position R^j of the center-of-mass, defined in such a way that $M\dot{R}^j = P^j$. Our strategy is to manipulate the local conservation statements of Eqs. (8.110) and (8.112) to obtain integral statements; the conserved integrals are then identified with M and P^j.

The total material mass of the fluid system is

$$m := \int \rho^*\, d^3x, \tag{8.132}$$

and this is conserved by virtue of Eq. (1) of Box 8.4: substituting $f=1$ gives $dm/dt = 0$ immediately.

To derive an expression for the total energy E we write Eq. (8.110) as

$$0 = \rho^*\frac{d}{dt}\Big(\frac{1}{2}v^2 + \Pi\Big) - \rho^* v^j\partial_j U + \partial_j(pv^j) + O(c^{-2}). \tag{8.133}$$

Integrating this over the volume occupied by the fluid, we find that the first term gives rise to

$$\frac{d}{dt}\int \rho^*\Big(\frac{1}{2}v^2 + \Pi\Big)\, d^3x$$

after taking the total time derivative outside the integral. By virtue of Eq. (8.120d) we find that the second term contributes

$$\frac{d}{dt}\int \rho^*\Big(-\frac{1}{2}U\Big)\, d^3x,$$

and the third term vanishes (by Gauss's theorem) after integration. Writing

$$E = \mathcal{T} + \Omega + E_{\rm int} + O(c^{-2}) \tag{8.134}$$

with

$$\mathcal{T} := \frac{1}{2}\int \rho^* v^2\, d^3x, \tag{8.135a}$$

$$\Omega := -\frac{1}{2}\int \rho^* U\, d^3x, \tag{8.135b}$$

$$E_{\rm int} := \int \rho^* \Pi\, d^3x, \tag{8.135c}$$

we have shown that $dE/dt = 0$. We recognize in $\mathcal{T}$ the total kinetic energy of the fluid, Ω is the total gravitational potential energy, and $E_{\rm int}$ is the total internal energy; these add up to the conserved energy E.

The total mass-energy M of the fluid system is defined by $M := m + E/c^2$. Combining Eqs. (8.132) and (8.134), this is

$$M := \int \rho^*\bigg[1 + \frac{1}{c^2}\Big(\frac{1}{2}v^2 - \frac{1}{2}U + \Pi\Big)\bigg]\, d^3x + O(c^{-4}), \tag{8.136}$$

and we have that $dM/dt = 0$. We have encountered the expression of Eq. (8.136) before, back in Sec. 7.3.2, in the context of the post-Minkowskian approximation. In Chapter 7 the total mass-energy was defined by $M := c^{-2}\int(-g)(T^{00} + t^{00}_{\rm LL})\, d^3x$, in terms of the fluid's energy-momentum tensor and the Landau–Lifshitz pseudotensor. In the present context, our expression for M was obtained by manipulating the fluid equations, and it is reassuring that we have complete consistency between the two approaches.

8.4.6 Conservation of momentum

More work is required to identify the total momentum P^j and show that $dP^j/dt = 0$. We return to Eq. (8.112) and examine the second group of post-Newtonian terms. We see that the terms involving the Newtonian potential can be grouped into

$$2\rho^* v_j \frac{dU}{dt} = 2\rho^* \frac{d}{dt}(U v_j) - 2\rho^* U \frac{dv_j}{dt}, \tag{8.137}$$

or

$$2\rho^* v_j \frac{dU}{dt} = 2\rho^* \frac{d}{dt}(U v_j) + 2U\partial_j p - 2\rho^* U \partial_j U + O(c^{-2}) \tag{8.138}$$

after inserting the Newtonian version of Euler's equation. The terms involving the vector potential can similarly be expressed as $-4\rho^*(dU_j/dt - v^k\partial_j U_k)$, and we find that the local statement of momentum conservation becomes

$$\begin{aligned} 0 &= \partial_t(\mu\rho^* v^j) + \partial_k(\mu\rho^* v^j v^k) + \partial_j p - \rho^*\partial_j U + \frac{2}{c^2}U\partial_j p \\ &\quad - \frac{1}{c^2}\rho^*\Big(\frac{3}{2}v^2 - U + \Pi + p/\rho^*\Big)\partial_j U + \frac{2}{c^2}\rho^*\frac{d}{dt}\Big(Uv_j - 2U_j\Big) \\ &\quad + \frac{4}{c^2}\rho^* v^k\partial_j U_k - \frac{1}{c^2}\rho^*\partial_j\Psi + O(c^{-4}), \end{aligned} \tag{8.139}$$

with μ defined by Eq. (8.113).

We next integrate this equation over the volume occupied by the fluid. Examining each term in turn, we find that the integral of the first term contributes

$$\frac{d}{dt}\int \mu\rho^* v^j\, d^3x,$$

but that the integrals of the second, third, and fourth terms vanish by virtue of Gauss's theorem and the identity of Eq. (8.120a). For the fifth term we get $-2c^{-2}\int p\partial_j U\, d^3x$ after integration by parts. Leaving the sixth term alone for the time being, we note that the seventh term becomes

$$\frac{2}{c^2}\frac{d}{dt}\int \rho^*(Uv_j - 2U_j)\, d^3x = -\frac{2}{c^2}\frac{d}{dt}\int \rho^* U v_j\, d^3x \tag{8.140}$$

after involvement of Eq. (8.120b). And finally, we obtain

$$\begin{aligned} 0 &= \frac{d}{dt}\int \mu\rho^* v^j\, d^3x - \frac{2}{c^2}\frac{d}{dt}\int \rho^* U v_j\, d^3x \\ &\quad - \frac{1}{c^2}\int \rho^*\Big(\frac{3}{2}v^2 - U + \Pi + 3p/\rho^*\Big)\partial_j U\, d^3x + \frac{4}{c^2}\int \rho^* v^k\partial_j U_k\, d^3x \\ &\quad - \frac{1}{c^2}\int \rho^*\partial_j\psi\, d^3x - \frac{1}{2c^2}\int \rho^*\partial_{ttj}X\, d^3x + O(c^{-4}) \end{aligned} \tag{8.141}$$

after inserting $\Psi = \psi + \frac{1}{2}\partial_{tt}X$ within the last term. This simplifies to

$$0 = \frac{d}{dt}\int \mu\rho^* v^j\, d^3x - \frac{2}{c^2}\frac{d}{dt}\int \rho^* U v_j\, d^3x - \frac{1}{2c^2}\int \rho^*\partial_{ttj}X\, d^3x + O(c^{-4}) \tag{8.142}$$

when we invoke the identities of Eqs. (8.120c) and (8.120e).

We must now work on the term involving the superpotential. We write

$$\rho^*\partial_{ttj}X = \rho^*\frac{d}{dt}\big(\partial_{tj}X\big) - \rho^* v^k\partial_{tjk}X \tag{8.143}$$

and integrate. We obtain

$$\int \rho^*\partial_{ttj}X\, d^3x = -\frac{d}{dt}\int \rho^* U_j\, d^3x + \frac{d}{dt}\int \rho^*\Phi_j\, d^3x \tag{8.144}$$

after using Eqs. (8.10) and (8.122), and we note that by virtue of Eq. (8.120b), the first integral on the right-hand side can also be expressed as $\int \rho^* U v_j\, d^3x$.

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Collecting results, we have obtained the conservation statement $dP^j/dt = 0$, with the total momentum P^j identified as

$$P^j := \int \rho^* v^j \biggl[1 + \frac{1}{c^2}\Bigl(\frac{1}{2}v^2 - \frac{1}{2}U + \Pi + p/\rho^* \Bigr) \biggr] d^3x - \frac{1}{2c^2} \int \rho^* \Phi^j \, d^3x + O(c^{-4}). \tag{8.145}$$

We recall that the potential Φ^j was defined by Eq. (8.8). In Chapter 7 the total momentum was defined as $P^j := c^{-1} \int (-g)(T^{0j} + t^{0j}_{\rm LL})\, d^3x$, and this led to the expression displayed in Eq. (7.65). A glance at Eq. (8.120g) confirms that the two expressions are equivalent.

The total momentum of a post-Newtonian spacetime can always be made to vanish by performing a post-Galilean transformation of the type described in Sec. 8.3.6. The transformation is characterized by the velocity vector $V^j = P^j/M$.

8.4.7 Center-of-mass

Inspection of Eq. (8.132) suggests that a plausible expression for the position of the center-of-mass might be

$$R^j := \frac{1}{M} \int \rho^* x^j \biggl[1 + \frac{1}{c^2}\Bigl(\frac{1}{2}v^2 - \frac{1}{2}U + \Pi \Bigr) \biggr] d^3x + O(c^{-4}). \tag{8.146}$$

This matches the result obtained back in Sec. 7.3.2 on the basis of the post-Minkowskian definition $R^j := (Mc^2)^{-1} \int (-g)(T^{00} + t^{00}_{\rm LL}) x^j \, d^3x$. We confirm this result by proving that with Eq. (8.146), we can produce the expected center-of-mass relation

$$M\dot{R}^j = P^j + O(c^{-4}). \tag{8.147}$$

Once P^j has been set equal to zero by performing a post-Galilean transformation, the position of the center-of-mass is fixed in space, and a constant translation of the spatial coordinates allows us to set $R^j = 0$. The conditions $P^j = 0$ and $R^j = 0$ define the *center-of-mass frame* of the fluid system.

We express Eq. (8.146) as $MR^j = \int \nu \rho^* x^j \, d^3x + O(c^{-4})$, with $\nu = 1 + c^{-2}(\frac{1}{2}v^2 - \frac{1}{2}U + \Pi)$. Differentiation with respect to time produces

$$M\dot{R}^j = \int \nu \rho^* v^j \, d^3x + \int \rho^* x^j \frac{d\nu}{dt} \, d^3x, \tag{8.148}$$

and this can be written in the form

$$M\dot{R}^j = P^j - \frac{1}{c^2} \int p v^j \, d^3x + \frac{1}{2c^2} \int \rho^* \Phi^j \, d^3x + \int \rho^* x^j \frac{d\nu}{dt} \, d^3x \tag{8.149}$$

after incorporating Eq. (8.145). The derivative of ν can be evaluated with the help of Euler's equation (8.104) and the first law of thermodynamics, Eq. (8.105). We obtain

$$\rho^* x^j \frac{d\nu}{dt} = -\frac{1}{c^2} x^j \partial_k (p v^k) - \frac{1}{2c^2} \rho^* x^j \bigl(\partial_t U - v^k \partial_k U \bigr) + O(c^{-4}), \tag{8.150}$$

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and integration yields

$$\int \rho^* x^j \frac{dv}{dt}\, d^3x = \frac{1}{c^2}\int p v^j\, d^3x - \frac{1}{2c^2}\int \rho^* \Phi^j\, d^3x + O(c^{-4}); \tag{8.151}$$

we integrated the first term by parts and made use of Eq. (8.120g) for the second term. Making the substitution into Eq. (8.149), we see that Eq. (8.147) is indeed satisfied.

8.5 Bibliographical remarks

The Maxwell-like formulation of the equations of post-Newtonian theory, as reviewed in Box 8.1, has received a number of presentations in the literature. One of the earliest incarnations was provided by Braginskii, Caves, and Thorne (1977).

The classic approach to post-Newtonian theory can be traced to the earliest days of general relativity. Representative works are Lorentz and Droste (1917), Eddington and Clark (1938), Einstein, Infeld, and Hoffmann (1938), and the treatise by Fock (1959). The work was invigorated by Chandrasekhar in the nineteen sixties, through a series of papers written with students and collaborators: Chandrasekhar and Contopoulos (1967), Chandrasekhar (1965 and 1969), Chandrasekhar and Nutku (1969), and Chandrasekhar and Esposito (1970).

The theory of post-Newtonian coordinate transformations developed in Sec. 8.3 was first initiated by Damour, Soffel, and Xu (1991); our treatment follows Racine and Flanagan (2005). The post-Galilean subclass of transformations was first investigated in Chandrasekhar and Contopoulos (1967). The rotating coordinates of Sec. 8.3 and the darned disk are described in some detail in Robertson and Noonan (1968).

The post-Newtonian theory of fluid dynamics was first developed in Chandrasekhar (1965 and 1969). Our treatment in Sec. 8.4 follows the master's work quite closely.

8.6 Exercises

8.1 Show that the inverse to the metric of Eqs. (8.20) is given by

$$\begin{aligned} g^{00} &= -1 - \frac{2}{c^2}U - \frac{2}{c^4}\big(\Psi + U^2\big) + O(c^{-6}), \\ g^{0j} &= -\frac{4}{c^3}U^j + O(c^{-5}), \\ g^{jk} &= \left(1 - \frac{2}{c^2}U\right)\delta^{jk} + O(c^{-4}), \end{aligned}$$

where $U^j := \delta^{jk}U_k$. Show that the metric determinant is $\sqrt{-g} = 1 + 2U/c^2 + O(c^{-4})$. Verify Eqs. (8.15) for the Christoffel symbols.

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8.2 Show that the post-Newtonian version of the geodesic equation $Du^\alpha/d\tau = 0$ can be presented in the form

$$\frac{d(g_s\gamma \boldsymbol{v})}{dt} = \gamma\Big(\boldsymbol{E}_\mathrm{g} + \boldsymbol{v}\times\boldsymbol{B}_\mathrm{g} + v^2\boldsymbol{\nabla} g_s\Big) + O(c^{-4}),$$

where $\boldsymbol{E}_\mathrm{g}$ and $\boldsymbol{B}_\mathrm{g}$ are the gravitational fields defined in Box 8.1, $\gamma := dt/d\tau$, and $g_s = 1 + 2c^{-2}U$ is the coefficient of the spatial part of the metric.

8.3 (a) Show that the coordinate transformation

$$t = \bar{t}, \qquad x^j = \bar{x}^j + \frac{\lambda}{c^2}\partial^{\bar{j}}\bar{X},$$

in which λ is a constant, produces a new post-Newtonian metric given by

$$\begin{aligned}
\bar{g}_{00} &= g_{00} - \frac{2\lambda}{c^4}(\bar{U}^2 - \bar{\Phi}_2 + \bar{\Phi}_W),\\
\bar{g}_{0j} &= g_{0j} + \frac{\lambda}{c^3}\partial_{\bar{t}\bar{j}}\bar{X},\\
\bar{g}_{jk} &= g_{jk} + \frac{2\lambda}{c^2}\partial_{\bar{j}\bar{k}}\bar{X},
\end{aligned}$$

where $g_{\alpha\beta}$ denotes the original post-Newtonian metric, with all potentials defined in terms of $\bar{x}^\alpha$, and where $\bar{\Phi}_W$ is an auxiliary potential (known as the Whitehead potential) defined by

$$\bar{\Phi}_W := G^2\int \bar{\rho}^{*\prime}\bar{\rho}^{*\prime\prime}\frac{(\bar{x}-\bar{x}')_j}{|\bar{\boldsymbol{x}}-\bar{\boldsymbol{x}}'|^3}\left[\frac{(\bar{x}'-\bar{x}'')^j}{|\bar{\boldsymbol{x}}-\bar{\boldsymbol{x}}''|} - \frac{(\bar{x}-\bar{x}'')^j}{|\bar{\boldsymbol{x}}'-\bar{\boldsymbol{x}}''|}\right]d^3\bar{x}'d^3\bar{x}''.$$

(b) Consider a static system with a "point mass" at the origin. This assumption allows us to ignore the potentials $\bar{\psi}$, $\bar{\Phi}_2$, and $\bar{\Phi}_W$, and to set all time derivatives to zero. Find the value of λ for which the metric is linear in $\bar{U}$ to post-Newtonian order. Show that the line element in spherical polar coordinates can be expressed to post-Newtonian order as

$$ds^2 = -(1 - R/\bar{r})(cd\bar{t})^2 + (1 - R/\bar{r})^{-1}d\bar{r}^2 + \bar{r}^2\big(d\bar{\theta}^2 + \sin^2\bar{\theta}\,d\bar{\phi}^2\big),$$

where $R = (2G/c^2)\int \rho^*\,d^3x = 2Gm/c^2$. What is this metric?

8.4 (a) Using the expressions for the Landau–Lifshitz pseudotensor given in Eqs. (7.48) and (7.49), together with the post-Newtonian expression for the potentials $h_2^{\alpha\beta}$ from Box 7.5, show that the components of the effective energy-momentum pseudotensor τ^{0j} and τ^{jk} are given to post-Newtonian order by

$$\begin{aligned}
c^{-1}\tau^{0j} &= \rho^* v^j\left[1 + \frac{1}{c^2}\left(\frac{1}{2}v^2 + 3U + \Pi + p/\rho^*\right)\right]\\
&\quad + \frac{1}{4\pi Gc^2}\Big[3\partial_t U\partial^j U + 4\big(\partial^j U^k - \partial^k U^j\big)\partial_k U\Big] + O(c^{-4}),
\end{aligned}$$

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and

$$
\begin{aligned}
\tau^{jk} &= \rho^* v^j v^k \biggl[1 + \frac{1}{c^2}\Bigl(\frac{1}{2}v^2 + 3U + \Pi + p/\rho^* \Bigr) \biggr] + p\Bigl(1 + \frac{2}{c^2}U\Bigr)\delta^{jk} \\
&\quad + \frac{1}{4\pi G}\Bigl(\partial^j U \partial^k U - \frac{1}{2}\delta^{jk}\partial_n U \partial^n U \Bigr) \\
&\quad \frac{1}{4\pi G c^2}\Bigl\{ 2\partial^{(j} U \partial^{k)} \Psi - 16 \partial^{[j} U^{n]} \partial^{[k} U^{n]} + 8 \partial^{(j} U \partial_t U^{k)} \\
&\quad - \delta^{jk}\Bigl[\partial^n U \partial^n \Psi - 4 \partial^{[m} U^{n]} \partial^{[m} U^{n]} + 4\partial^n U \partial_t U^n + \frac{3}{2}(\partial_t U)^2 \Bigr] \Bigr\} \\
&\quad + O(c^{-4}),
\end{aligned}
$$

where $\Psi = \psi + \frac{1}{2}\partial_{tt} X$.

(b) Show that the conservation statement $\partial_\beta \tau^{j\beta} = 0$ yields the post-Newtonian version of Euler's equation, as displayed in Eq. (8.119). You may make use of the continuity equation for ρ^*, the first law of thermodynamics, and the Poisson equations satisfied by the various potentials.

8.5 In this problem we consider the equilibrium structure of a spherical body in post-Newtonian theory, thereby generalizing the Newtonian discussion of Sec. 2.2. We assume that the matter distribution is static and spherically-symmetric, so that all variables depend on r only. Show that under these conditions, the gravitational potentials are determined by the set of equations

$$
\frac{dU}{dr} = -\frac{Gm}{r^2}, \qquad \frac{dm}{dr} = 4\pi r^2 \rho^*,
$$

$$
\frac{d\Psi}{dr} = -\frac{Gn}{r^2}, \qquad \frac{dn}{dr} = 4\pi r^2 \rho^*(-U + \Pi + 3p/\rho^*),
$$

where n is a post-Newtonian auxiliary variable analogous to the Newtonian mass function m. Show also that the equation of hydrostatic equilibrium becomes

$$
\frac{dp}{dr} = -\frac{G\rho^*}{r^2}\biggl\{ m + \frac{1}{c^2}\Bigl[(-3U + \Pi + p/\rho^*)m + n \Bigr] \biggr\} + O(c^{-4}).
$$

These equations are to be supplemented by an equation of state $p = p(\rho^*)$ and the first law of thermodynamics, $d\Pi = (p/\rho^{*2})\,d\rho^* + O(c^{-2})$.

8.6 The equations derived in the preceding problem should agree with the exact formulation of the structure equations, as presented in Sec. 5.6.5, when these are expressed as a post-Newtonian expansion. The comparison, however, is not entirely straightforward, because the formulations use different variables and coordinates.

(a) By comparing the metric of Eq. (5.156) with the post-Newtonian metric of Eq. (8.2), relate the radial coordinate $\bar{r}$ of Sec. 5.6.5 (perversely denoted r there) to the harmonic radial coordinate r employed in this chapter.

(b) Show that the metric functions are related by $\Phi = U + \Psi/c^2 + O(c^{-4})$ and $\bar{m} = m + O(c^{-2})$, where $\bar{m}$ is the mass function defined by Eq. (5.158) (and denoted m there).

(c) Prove that when Eqs. (5.215), (5.216), and (5.218) are expanded through order c^{-2} in a post-Newtonian expansion, they agree with the equations derived in the preceding problem.

(d) Show that the comparison relies on the identification

$$\bar{m} = m + \frac{1}{c^2}\big(n + mU - Gm^2/r - 4\pi r^3 p\big) + O(c^{-4})$$

for the relativistic mass function.

8.7 Using the Newtonian Euler equation and the first law of thermodynamics, verify the energy conservation equation Eq. (8.110).

8.8 Show that the conserved angular momentum tensor for an isolated system is given to post-Newtonian order by

$$J^{jk} = 2\int \rho^* x^{[j}\bigg\{v^{k]} + \frac{v^{k]}}{c^2}\bigg(\frac{1}{2}v^2 + 3U + \Pi + p/\rho^*\bigg) - \frac{1}{c^2}\bigg(4U^{k]} - \frac{1}{2}\partial_t^{k]}X\bigg)\bigg\}d^3x\,.$$

8.9 Verify the integral identities of Eqs. (8.120e) and (8.120f).